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COUNTING PSEUDO-HOLOMORPHIC SUBMANIFOLDS IN DIMENSION 4

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The purpose of this article is to describe a certain invariant (called the Gromov invariant) for compact symplectic 4-manifolds which assigns an integer to each dimension 2-cohomology class. Roughly speaking, the invariant counts, with suitable weights, compact, pseudo-holomorphic submanifolds whose fundamental class is Poincaré dual to the cohomology class in question.

A version of this invariant was introduced originally by Gromov [2] to study the deformation classes of symplectic structures on manifolds with the homology of $\mathbb{CP}^2$. Subsequently, Ruan [7] extended Gromov's constructions to all symplectic 4-manifolds; the generalization of Ruan counts only connected, pseudo-holomorphic submanifolds. The invariant described below generalizes the construction of Ruan. The definition was sketched in [10] where the invariant was identified with the Seiberg-Witten invariants [13] of the symplectic manifold. However, the definition in [10] is incomplete in one respect - in its description of counting weights for multiply covered tori with trivial normal bundle. (The discussion in [7] is erroneous in this regard.) Thus, this article also serves to clear up any confusion stemming from counting these multiply covered tori.

Note that the equivalence claimed in [10] between the Seiberg- Witten invariant and the Gromov invariant as defined here holds for manifolds with $b^{2+} > 1$. The details of the proof will appear shortly (see [11], [12]).

This article is organized as follows: Section 1 defines the Gromov invariant as a weighted count of pseudo-holomorphic submanifolds. (See

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Theorem 1.1.) The definition in Section 1 is complete modulo two auxilliary definitions. The first of these describes the counting weight to give a connected component of some pseudo-holomorphic submanifold. These weights are described in Sections 2 and 3, first for non-multiply covered components, and then for multiply covered, toroidal components. As is usually the case in these matters, the counting can be done most efficiently when the almost complex structure is an appropriately generic one. The definition of "appropriately" is the other missing piece from Section 1 and is given in Section 4.

The proof of Theorem 1.1 and various other assertions in Sections 1-4 are deferred to Section 5. The sixth section provides an example which demonstrates the necessity of some of the complications in the counting definition for multiply covered, pseudo-holomorphic tori from Section 3. (The example in Section 6 exhibits one of the problems with the definition in [10].) The final section is here for fun; it provides a localization proof of the Riemann-Roch formula for curves by exploiting a certain $\mathbb{C}$-antilinear perturbation of the Cauchy-Riemann operator which plays a major role in the earlier sections.

1. The definition of Gr

In all that follows, X denotes a compact, connected 4-dimensional manifold with a symplectic form, denoted by ω. The purpose of this section and the next two sections is to define a map

$$\mathrm{Gr} : H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z} \tag{1.1}$$

whose value depends only on the form ω up to continuous homotopies through symplectic forms. The definition of Gr in (1.1) is a two-step affair. The first step, below serves as a digression to introduce the reader to some basic facts about symplectic 4-manifolds.

Step 1. Because ω is non-degenerate, the 4-form $\omega \wedge \omega$ orients X. This orientation will be implicit in what follows.

Let J denote an almost complex structure for TX which is compatible with the form ω. Thus, J is a section of $\mathrm{End}(TX)$ whose square is minus the identity. And, the bilinear form $g \equiv \omega(\cdot, J(\cdot))$ on TX defines a Riemannian metric. The almost complex structure splits $TX \otimes \mathbb{C}$ as $T_{1,0} \oplus T_{0,1}$ where the former consists of the holomorphic tangent vectors, i.e., vector of the form $v - i \cdot Jv$ where v is in TX. The complexified cotangent bundle, $T^*X \otimes \mathbb{C}$ splits analogously as $T^{1,0} \oplus T^{0,1}$. Note that the canonical bundle of X is, by definition, the complex line bundle $K \equiv \det(T^{1,0})$.

Given J, one can introduce the notion of a pseudoholomorphic submanifold in X. Call a compact submanifold $\Sigma \subset X$ pseudo-holomorphic when J maps $T\Sigma$ to itself as a subspace of $TX|_\Sigma$. Note that such a pseudo-holomorphic submanifold is automatically symplectic and so inherits a natural orientation. In particular, the fundamental class, $[\Sigma]$, is non-zero and of infinite order in $H_2(X;\mathbb{Z})$.

A connected, pseudo-holomorphic submanifold Σ has the special property that its genus is determined apriori by the class $[\Sigma]$ via the adjunction formula:

$$\text{genus}(\Sigma) = 1 + \frac{1}{2}(e \cdot e + c \cdot e). \tag{1.2}$$

Here, e denotes the Poincaré dual to $[\Sigma]$, and c denotes the first Chern class of K. The notation $\cdot$ signifies the cup product pairing on the two dimensional cohomology. (Thus, $e \cdot e$ is equal to the Euler number of the normal bundle of Σ when the latter is oriented in the natural way.)

By the way, since almost complex structures on 2-dimensional manifolds are always integrable, Σ inherits from X the structure of a complex holomorphic curve. As a complex curve, the embedding of Σ into X is a pseudo-holomorphic map in the sense used by Gromov [2]. (A smooth map φ from a complex curve Σ into X is called pseudo-holomorphic when φ intertwines the almost complex structure on Σ with that on X.) In any event, the complex structure on Σ which is induced by its embedding in X will be implicit in much of what follows.

Step 2. Fix $e \equiv H^2(X;\mathbb{Z})$. First, introduce the integer

$$d = d(e) = \frac{1}{2}(e \cdot e - c \cdot e). \tag{1.3}$$

(This is an integer because $c \cdot e = e \cdot e \mod (2)$ for all classes e.)

If the integer d in (1.3) is positive, choose a set $\Omega \subset X$ of d distinct points. Now, introduce the set $\mathcal{H} \equiv \mathcal{H}(e, J, \Omega)$ whose elements are the (unordered) sets of pairs $\{(C_k, m_k)\}$ of disjoint, connected, pseudo-holomorphic submanifold $C_k \subset X$ and positive integer m_k, which are constrained as follows:

(1.4)

1. Let $e_k \in H^2(X;\mathbb{Z})$ denote the Poincaré dual of the fundamental class of C_k. Introduce d_k as in (1.3) and require that $d_k \geq 0$.

2. If $d_k > 0$, then C_k contains precisely d_k members of the set Ω.

3. The integer $m_k = 1$ unless C_k is a torus with trivial normal bundle. This happens if and only if $e_k \cdot e_k = c \cdot e_k = 0$.

4. $\sum_k m_k e_k = e$.

By the way, note that when $h \equiv \{(C_k, m_k)\} \in \mathcal{H}$, the various C_k are pairwise disjoint. This implies that the classes $\{e_k\}$ are pairwise orthogonal with respect to the cup product pairing. (Pseudo-holomorphic submanifolds can have only positive local intersection number [2], [4].) And, this mutual orthogonality implies that

$$\sum_k d_k = d. \tag{1.5}$$

With $\mathcal{H}$ understood, remark that the value of the Gromov invariant for e is defined below by making a weighted count of the elements of $\mathcal{H}$ when using a suitably generic choice for J and Ω. In this regard, note that the set $\mathcal{H}(e, J, \Omega)$ is finite for a suitably generic choice of (J, Ω); this assertion is made precise below. Furthermore, for a suitably generic choice of (J, Ω), the count of elements in $\mathcal{H}(e, J, \Omega)$ is obtained by weighing each $h \equiv \{(C_k, m_k)\} \in \mathcal{H}(e, J, \Omega)$ by a product of integers

$$q(h) = \prod_k r(C_k, m_k). \tag{1.6}$$

Here, $r(\cdot)$ is an integer assignment to pairs (C, m) of positive integer m and pseudo-holomorphic submanifold $C \subset X$.

Suppose that $d \geq 0$. With $r(\cdot)$ as given below (in (2.13) and Definition 3.2), the Gromov invariant for e is defined to be

$$\mathrm{Gr}(e) = \sum_h q(h), \tag{1.7}$$

where the sum is over elements $h \in \mathcal{H}(e, J, \Omega)$, and $\mathcal{H}(e, J, \Omega)$ is defined with respect to a suitably generic choice of ω-compatible, almost complex structure J on X and set Ω of distinct of d (when $d > 0$) points in X. (This is the set of admissable (J, Ω) as characterized in Definition 4.2 and Proposition 4.3, below.)

When d in (1.3) is negative, set

$$\mathrm{Gr}(e) = 0. \tag{1.8}$$

Given these specifications, consider:

Theorem 1.1. *Let X be a compact 4-manifold with symplectic form ω. Fix a class $e \in H^2(X;\mathbb{Z})$, and use (2.13) and Definition 3.2 to specify $r(\cdot)$. Then, compute (1.7) with $\mathcal{H}(e, J, \Omega)$ defined from an admissable pair (J, Ω) as described in Definition 4.2 and Proposition 4.3. The resulting assignment of integers to classes in $H^2(X;\mathbb{Z})$ defines a map, $\mathrm{Gr}(\cdot)$, which is independent of the precise choice of (J, Ω). Furthermore if ω_1 is a second symplectic form on X, then the resulting $\mathrm{Gr}(\cdot)$ maps agree when there is a path of symplectic forms in $C^\infty(X, \Lambda^2 T^*)$ which begins at ω and ends at ω_1. Finally, if φ is a diffeomorphism of X, then $\mathrm{Gr}(\varphi^*(\cdot))$ as computed with $\varphi^*\omega$ is the same as $\mathrm{Gr}(\cdot)$ as computed with ω.*

The proof of this theorem is given in Section 5.

There is a natural generalization of this Gromov invariant which was introduced in a physics context by Witten [14] (see, also [5], [8]). This generalization takes the form of a map

$$\mathrm{GW} : H^2(X;\mathbb{Z}) \to \Lambda^* H^1(X;\mathbb{Z}), \tag{1.9}$$

which has the property that $\mathrm{GW}(e)$ projects to the summand $\mathbb{Z} \in \Lambda^*$ as $\mathrm{Gr}(e)$. (Remember that $\Lambda^0 \approx \mathbb{Z}$.)

The definition of $\mathrm{GW}(e)$ occupies the three steps that follow.

Step 1. Let $d = d(e)$. Then $\mathrm{GW}(e)(\cdot)$ annihilates $\Lambda^p H_1(X;\mathbb{Z})$ when $p > 2 \cdot d$ or when p is odd.

Step 2. Fix an integer $n \in \{0, \ldots, d(e)\}$ and an ordered set $\Gamma = \{\gamma_1, \ldots, \gamma_{2n}\}$ of disjoint, oriented, 1-dimensional submanifolds in X. Also, fix a set $\Omega \subset X$ of $d - n$ distinct points. Now introduce the set $\mathcal{H} = \mathcal{H}(e, J, \Gamma, \Omega)$ whose typical element, h, consists of an unordered set of triples $\{(C_k, m_k, \Gamma_k)\}$ for which a given triple (C_k, m_k, Γ_k) consists of a compact, pseudo-holomorphic submanifold C_k, a positive integer m_k

and an unordered subset $\Gamma_k \subset \Gamma$ of an even number of elements. Each component (C_k, m_k, Γ_k) of h is further constrained as follows:

(1.10)

1. For each k, define d_k as in (1.4.1), and require $d_k \geq 0$.
2. For each k, let $2n_k$ denote the number of elements of Γ_k. Then $0 \leq n_k \leq d_k$,
3. For each k, C_k contains precisely $d_k - n_k$ points of Ω, and intersects each member of Γ_k exactly once.
4. If $k \neq k'$, then Γ_k is disjoint from $\Gamma_{k'}$; but require that $\cup_k \Gamma_k = \Gamma$.
5. If $k \neq k'$, then C_k is disjoint from $C_{k'}$.
6. For each $k, m_k = 1$ unless C_k is a torus with trivial normal bundle in which case $m_k \geq 1$ is allowed.
7. $\sum_k m_k \cdot e_k = e$.

Step 3. For each j, let $[\gamma_j]$ denote the equivalence class of γ_j in $H_1(X; \mathbb{Z})/\text{Torsion}$. Then set $\phi = [\gamma_1] \wedge [\gamma_2] \wedge \ldots \wedge [\gamma_{2n}]$. Theorem 1.2, below, asserts that for a suitably generic choice of (J, Ω, Γ), the set $\mathcal{H}$ is finite. In particular, for a suitably generic choice of (J, Ω, Γ), the value of $\text{GW}(e)(\phi)$ can be computed by counting the members of $\mathcal{H}$ in the appropriate way. This count for $\text{GW}(e)(\phi)$ is given by (1.7) where $q(h)$ is computed via

$$q(h) = (-1)^{o(h)} \textstyle\prod_k r(C_k, m_k, \Gamma_k). \tag{1.11}$$

Here, $o(h) \in \{0, 1\}$, and each $r(C_k, m_k, \Gamma_k)$ is an integer. Note that $o(h)$ is defined at the end of the next section (see (2.14).) Meanwhile, $r(C, m, \emptyset) = r(C, m)$, where the latter is defined in (2.13) or Definition 3.2. Finally, $r(C, 1, \Gamma)$ for $\Gamma \neq \emptyset$ is defined in (2.15).

Theorem 1.2. *Let X be a compact 4-manifold with symplectic form ω. Fix a class $e \in H^2(X; \mathbb{Z})$. Given $n \in \{0, \ldots, d(e)\}$, fix classes $[\gamma_1], \ldots, [\gamma_{2n}] \in H_1(X; \mathbb{Z})/\text{Torsion}$. There is a Baire set of triples consisting of an ω-compatible, almost complex structure J on X, a set Ω of $d(e) - n$ distinct points in X and a set $\Gamma = \{\gamma_1, \cdots, \gamma_{2n}\}$*

oriented, 1-dimensional submanifolds in X with each γ_i in the corresponding equivalence class $[\gamma_i]$. Choose (J, Ω, Γ) from this Baire set and then $\mathcal{H}(e, J, \Omega, \Gamma)$ has finitely many elements. Furthermore, if $h = \{(C_k, m_k, \Gamma_k)\} \in \mathcal{H}$, then (2.13), (2.15) and Definition 3.2 define $r(\cdot)$ in (1.11), and (2.14) defines $o(h)$ in (1.11). Thus, $q(h)$ is well defined and so is $\mathrm{GW}(e)(\phi) = \sum_h q(h)$. *Then the following hold:*

- $\mathrm{GW}(e)(\phi)$ *is independent of the chosen data (J, Ω, Γ) from the Baire set and depends only on e, ϕ and the symplectic form ω up to continuous deformations through symplectic forms.*
- *The assignment of* $\mathrm{GW}(e)(\phi)$ *to ϕ extends as well defined homomorphism from* $\Lambda^n(H_1(X;\mathbb{Z})/\,\mathrm{Torsion})$ *to* $\mathbb{Z}$.
- $\mathrm{GW}(e)(1) = \mathrm{Gr}(e)$, *where the latter is described in Theorem 1.1.*
- *The resulting map* $\mathrm{GW} : H^2(X;\mathbb{Z}) \to \Lambda^*(H^1(X;\mathbb{Z})$ *is naturally equivariant under the action of* $\mathrm{Diff}(X)$.

The proof of Theorem 1.2 is given at the end of Section 5.

2. The definition of $r(C, 1)$

Let C be a compact, connected, pseudo-holomorphic submanifold of X. If C is non-degenerate in a technical sense which is defined below, then

$$r(C, 1) = \pm 1. \tag{2.1}$$

The purpose of this section is to explain how one determines the sign. This task is accomplished in two steps.

Step 1. This step introduces a notion of nondegeneracy for a connected, pseudo-holomorphic submanifold C. A three part digression is required for this purpose.

Part 1 of the digression defines a structure of a holomorphic vector bundle on the normal bundle to C in X. To start the definition, remark that metric g (as defined above) splits TX along C as

$$TX|_C = TC \oplus N_C \tag{2.2}$$

with each summand being J-invariant. Here, N_C is the normal bundle to C in X.

Since $J|_C$ preserves (2.2), the real 2-plane bundle N_C can be thought of as a complex line bundle over X. The latter will be denoted by N to distinguish these two ways of viewing the normal bundle of X.

The bundle N inherits the structure of a holomorphic vector bundle over C. The point here is that the Levi-Civita connection for the metric g induces a connection on N_C which decomposes $T(N_C)$ as $p^*(TC \oplus N_C)$. Here, $p : N_C \longrightarrow C$ is the bundle projection. But, as remarked, the almost complex structure $J|_C$ preserves the splitting in (1.4) and so its pull-back to $T(N_C)$ defines an almost complex structure, J_0, on TN_C which is invariant under translations along the fiber and preserves the connection splitting.

The almost complex structure J_0 is integrable and so N_C has the structure of a complex manifold. This is to say that J_0 gives the complex line bundle N the structure of a holomorphic vector bundle over the complex curve C. Note in particular, that the fibers of N are J_0-holomorphic lines in N.

Part 2 of the digression defines an exponential map of sorts with which to pull the almost complex structure J from X back to a neighborhood, $U \subset N_C$, of the zero section. In particular, the implicit function theorem (see, e.g. Lemmas 5.4 and 5.5 in [11]) can be used to find a disk bundle $U \subset N_C$ together with a particularly nice embedding $\varphi : U \longrightarrow X$ which enjoys the following properties:

(2.3)

1. φ maps the zero section as the identity on C.
2. The differential of φ along C is an isometry.
3. φ restricts to each fiber of U as a pseudo-holomorphic map. (Use J_0 to define the complex structure on the fibers of U.)

The map φ pulls back the almost complex structure J to U. This pull-back will also be denoted by J.

Now U, as a subbundle of N_C, also inherits the complex structure J_0. In general, J_0, differs from J. However, one can write

$$J = J_0 + T, \tag{2.4}$$

where T is a section over U of $\mathrm{Hom}(p^*T_C, p^*TN_C)$ which vanishes along the zero section.

Of particular interest here is the evaluation at the zero section of the derivative of T along the fibers of U. The latter defines, via orthogonal projections, sections ν of $T^{0,1}C$ and μ of $T^{0,1}C \otimes N^{\otimes 2}$. To be precise here, consider a local coordinate system about a point $p \in C$. Let z be a local holomorphic parameter for C near p. Trivialize the complex line bundle N near C and let η be the (complex) fiber coordinate. This can be done so that the J_0 version of $T^{1,0}(N_C)$ is given by the span of $\{dz, d\eta + a \cdot \eta \cdot d\bar{z}\}$. Here, $\bar{z}$ is the complex conjugate of z and a is a complex valued function near p. With the preceding understood, consider that the almost complex structure J on U defines a different $T^{1,0}$, the latter being the span of

$$\{dz + k \cdot d\bar{z}, d\eta + h \cdot d\bar{z}\}, \tag{2.5}$$

where k and h are complex valued functions on the restriction of D to a fiber bundle over a neighborhood of p which vanish where η is zero. Taylor's theorem then writes h as

$$h = (a + \nu) \cdot \eta + \mu \cdot \bar{\eta} + \mathcal{O}(|\eta|^2). \tag{2.6}$$

The complex valued functions ν and μ are apriori defined on a neighborhood of p in C, but they extend to the whole of C as respective sections of $T^{0,1}C$ and of $T^{0,1}C \otimes N^{\otimes 2}$.

As a parenthetical remark, note that one can rightfully say that J is <u>integrable</u> to the first order along C when $\mu \equiv 0$ since this is a necessary and sufficient condition for the torsion tensor of J to vanish along C.

Part 3 of the digression defines a differential operator D on the space of sections over C of the real 2-plane bundle N_C. This operator sends a section of N_C to a section of the underlying real 2-plane bundle of the complex bundle $N \otimes T^{1,0}C$. The operator D takes a section of N (thought of as a section of N_C) and gives

$$Ds \equiv \bar{\partial} s + \nu s + \mu \bar{s}, \tag{2.7}$$

thought of as a section of the underlying real bundle of $N \otimes T^{1,0}C$. Here, $\bar{\partial}$ is the usual d-bar operator on sections of N, while $\bar{s}$ denotes the image of s in N^{-1} under the tautological $\mathbb{C}$-anti-linear isomorphism between N and N^{-1}.

There are two important features of this operator D to keep foremost in mind: First, the operator D is <u>canonically</u> associated to the pseudo-holomorphic submanifold C. Second, the operator D is an elliptic operator. (It is evidently a zero'th order perturbation of the standard

d-bar operator.) In particular, the kernel and cokernel of D are both finite dimensional vector spaces.

Remark that kernel of the operator D should be thought of as giving a sort of Zariski tangent space to the space of pseudo-holomorphic embedding of C in X as a point in the space of all smooth embeddings of C into X.

Now, when $\Omega \subset X$ is a discrete set of points, introduce $ev_\Omega : C^\infty(N) \longrightarrow (\oplus_{p\in\Omega} N|_p)$ to denote the evaluation map.

Definition 2.1. Let $e \in H^2(X;\mathbb{Z})$ and if $d = d(e) > 0$, fix a set Ω of d distinct points in X. Let $C \subset X$ be a connected, pseudo-holomorphic submanifold containing Ω. If $d = 0$, call C non-degenerate when cokernel$(D) = \{0\}$. When $d > 0$, call C non-degenerate when the operator

$$D \oplus ev_\Omega : C^\infty(N) \longrightarrow C^\infty(N \otimes T^{0,1}C) \oplus (\oplus_{p\in\Omega} N|_p). \tag{2.8}$$

has trivial cokernel.

Note that in the $d \geq 0$ case, the operator in (2.8) is Fredholm with index zero, and further note that D has trivial cokernel if the operator in (2.8) has trivial cokernel.

Step 2. Fix a set Ω of d points in X. This step defines the sign for $r(C, 1)$ in (2.1) when C is a non-degenerate, connected, pseudo-holomorphic submanifold of X which contains Ω.

The $\pm$ sign for a given such C is formally the sign of the determinant of the operator in (2.8). Formalities aside, the sign in question is defined as follows: First, fix a smooth map $\tau : [0, 1] \longrightarrow C^\infty(\mathrm{Hom}(N; T^{0,1}C))$ which vanishes at $t = 0$. Use τ_t to denote the homorphism $\tau|_t$. Then, for each $t \in [0, 1]$, define the operator D_t to take a section s of N to

$$D_t s \equiv (\overline{\partial} s + (\nu + \tau_t)s + t \cdot \mu\bar{s}, s(p_1), \ldots s(p_d)), \tag{2.9}$$

which is a point in $C^\infty(N \otimes T^{0,1}C) \oplus (\oplus_{p\in\Omega} N|_p)$. Note that D_t can be interpreted in a natural way as an index zero, Fredholm operator. Thus, its kernel and cokernel are finite dimensional with the same dimension. Second, note that the $t = 0$ operator D_0 is $\mathbb{C}$-linear. For a suitably generic choice of τ_0, both the kernel and cokernel of D_0 will be empty. With this understood, write

$$\mathrm{sign}(\det(D_0)) \equiv +1. \tag{2.10}$$

Second, remark that for a suitably generic choice of τ (a choice from a certain Baire (in particular, dense) subset of the apropriate Frechet space of smooth maps), the set of t for which D_t has non-empty kernel is finite; a set with some $N \geq 0$ elements. Also, at each such $t, \dim(\text{kernel}(D_t)) = 1$ and a certain linear map

$$M_t : \text{kernel}(D_t) \longrightarrow \text{cokernel}(D_t) \tag{2.11}$$

is an isomorphism. This M_t takes an element $s \in \text{kernel}(D_t)$ and assigns to it the projection into cokernel(D_t) of

$$M_t \equiv \left(\frac{d}{dt}\tau|_t\right) s + \mu\bar{s}. \tag{2.12}$$

These assertions can be proved using basic properties of analytic perturbation theory from [3].

With the preceding understood, choose the map τ from the aforementioned generic set and define $r(C, 1)$ (the sign of $\det(D)$) for the given C to equal

$$\text{sign}(\det(D)) \equiv (-1)^N. \tag{2.13}$$

Analytic perturbation theory (again from [3]) can be used to prove that the sign in (2.13) is independent of the choice of the map τ which is used in its definition.

Now consider the definition of $o(h)$ and $r(C_k, m_k, \Gamma_k)$ which appear in (1.11). The definition of $o(h)$ comes first. In this regard, remember that Γ is an ordered set which consists of $2n$ elements, $\{\gamma_1, \dots, \gamma_{2n}\}$. Meanwhile, $\{\Gamma_k\}$ partitions Γ. Agree to order the elements in each Γ_k by increasing index. (For example, if the unordered set $\Gamma_k = \{\gamma_5, \gamma_2\}$, then the ordering is $\{\gamma_2, \gamma_5\}$.) With each subset Γ_k ordered, reorder Γ by putting the ordered set Γ_1 first, then Γ_2, etc. This new ordering of Γ differs from the original one by a permutation, π. Set

$$o(h) = 0 \text{ or } 1 \text{ if } \pi \text{ is even or odd, respectively.} \tag{2.14}$$

Note that the fact that each Γ_k contains an even number of elements insures that the definition of $o(h)$ is insensitive to any relabeling of the sets $\{\Gamma_k\}$.

Now turn to the definition of $r(C_k, m_k, \Gamma_k)$. As remarked previously, $r(C_k, m_k, \emptyset)$ is the same as $r(C_k, m_k)$ as defined in (2.13) in the case $m_k = 1$, or else as defined in Definition 3.2 in the case where $m_k > 1$. Furthermore, the case $m_k > 1$ arises only when C_k is a torus with trivial normal bundle, and in this case $d_k = 0$ so $\Gamma_k = \emptyset$. Thus, it is necessary to consider here only $r(C_k, 1, \Gamma_k)$ when $\Gamma_k \neq \emptyset$. The definition here requires three steps.

Step 1. If J is suitably generic, then the operator D for C has kernel dimension d_k and cokernel dimension 0. Furthermore, kernel(D) is canonically oriented. (The orientation can be obtained by considering, as previously, a continuous family, $\{D_t\}_{0\leq t\leq 1}$ of deformations of D by zero'th order operators such that $D_1 = D$ and such that D_0 is complex linear. Because $d_k \geq 1$, the family can be chosen so that cokernel(D) = $\{0\}$ for each t. Since D_0 is complex linear, its kernel inherits a natural orientation, and then, by continuity, so does kernel(D).)

Step 2. Order the elements of Γ_k by increasing index. For the sake of argument, suppose, after ordering, $\Gamma_k = \{\gamma_1, \ldots, \gamma_{2p}\}$. Each γ_α can be perturbed slightly, if necessary, so that its intersection point with C_k is unchanged, but so that its tangent space at this intersection point is not a subline in the tangent space to C_k. With this understood, the tangent space to each γ_α at its intersection point with C_k defines a line in the normal bundle fiber. Let 1_α denote the quotient of the said normal bundle fiber by $T\gamma_\alpha$. Note that each l_α is oriented (since both $T\gamma_\alpha$ and the normal bundle are.) Then, $V = \oplus_{1\leq\alpha\leq 2p} 1_\alpha$ is naturally oriented, as an ordered direct sum of oriented lines.

Step 3. By assumption, C_k also contains a set Ω' of $d_k - p$ points of Ω. For each point $z \in \Omega'$, let $N|_z$. denote the fiber of the normal bundle at z. Then, the obvious evaluation map (at the points of Ω' and the intersection points of the elements in Γ_k with C_k) defines a linear map $H : \text{kernel}(D) \to (\oplus_{z\in\Omega'} N|_z) \oplus (\oplus_{1\leq\alpha\leq 2p} 1_\alpha)$. As will be argued in Section 5, when (J, Ω, Γ) is chosen from Theorem 1.2's Baire set, this map will be an isomorphism between oriented vector spaces. Thus, its determinant has a well defined sign. Set

$$r(C_k, 1, \Gamma_k,) = \text{sign}(\det(H)). \tag{2.15}$$

3. The definition of $r(C, m)$ when $m > 1$

The purpose of this section is to define $r(C, m)$ in (1.6) when m is larger than 1. This definition is a four step affair.

Step 1. In the case $m > 1$, the sign of det(D) from (2.13) enters also, but here it is only part of the story since $r(\cdot)$ in the case $m > 1$ is defined in terms of spectral data for D and also from twists of D by the non-trivial, real line bundles over C. To set the stage for the details, assume, for the balance of this section, that C is an embedded, pseudo-holomorphic torus in X with trivial normal bundle.

The four isomorphism classes of real line bundles on C are distinguished by their first Stieffel-Whitney classes in $H^1(C;\mathbb{Z}/2)$. Thus, given $\iota \in H^1(C;\mathbb{Z}/2)$, let ϵ_ι denote the corresponding real line bundle whose first Stieffel-Whitney class is ι. Let $C^\infty(\iota;N)$ denote the space of sections of $\epsilon_\iota \otimes N_C$. Then, D in (2.7) also can be viewed as an $\mathbb{R}$-linear operator on $C^\infty(\iota;N)$. The latter operator will be denoted as D_ι, with D used solely to denote the operator in (2.7) for the original $\iota = 0$ case.

Prior to defining $r(\cdot)$ it is necessary to establish certain spectral properties of the $\{D_\iota\}$. The key lemma is given below. The statement of the lemma introduces the notion of a subvariety in an infinite dimensional Frechet space. For the purposes below, such a subset, $\mathcal{D}$, is a countable, disjoint union $\cup\mathcal{D}^k$ of submanifolds with two properties: First, for each $k, \mathcal{D}^k$ has codimension k. Second, for each k_0, the union $\cup_{k\geq k_0}\mathcal{D}^k$ is closed. (This definition is, perhaps, not standard as it makes no reference to any analytic structure.)

Lemma 3.1. *Let C be a complex torus and N be a complex, holomorphic line bundle over C. Given a pair (ν,μ) of sections of $T^{0,1}C$ and $T^{0,1}C \otimes N^{\otimes 2}$, use (2.7) to define the $\mathbb{R}$-linear operator D on the space of sections of N. Given $\iota \in H^1(C;\mathbb{Z}/2)$, define the operator D_ι by (2.7) as an $\mathbb{R}$-linear operator on the space of sections of $\epsilon_\iota \otimes N$.*

1. *Consider the operator D_ι in (2.7) in the case $\mu \equiv 0$. In this case, the kernel of D_ι has real dimension either 0 or 2. Furthermore, there is a codimension-2 submanifold $\mathcal{D}_{0\iota} \subset C^\infty(C;T^{0,1}C)$ which is characterized by the fact that when $\nu \in \mathcal{D}_{0\iota}$ and D_ι is defined from ν and $\mu \equiv 0$, then* $\text{kernel}(D_\iota) \neq \emptyset$.

2. *Consider now the operator D_ι in (2.7) in the general case. Then the kernel of D_ι has real dimension either 0, 1 or 2. And, there is a real analytic subvariety $\mathcal{D}_\iota \subset C^\infty(C;T^{0,1}C \oplus (T^{0,1}C \otimes N^{\otimes 2}))$ with codimension 1 or greater strata that is characterized by the fact that when $(\nu,\mu) \in \mathcal{D}_\iota$ and D_ι is defined from (ν,μ), then* $\text{kernel}(D_\iota) \neq \emptyset$. *Note from Part 1 that $\mathcal{D}_\iota$ intersects the subspace of pairs (ν,μ) with $\mu = 0$ in codimension 2. Finally, when $\iota \neq \iota'$, then the intersection of $\mathcal{D}_\iota$ with $\mathcal{D}_\iota'$ is a subvariety whose top strata has codimension at least two.*

3. *The path components of $\mathcal{D}_\iota$ are labeled by $\mathbb{Z} \times \mathbb{Z}$ and each path component of $\mathcal{D}_\iota$ intersects a path component of $\mathcal{D}_{0\iota}$.*

Lemma 3.1 is proved in Section 5. The third assertion is not used below.

Set $\mathcal{D} \equiv \cup_\iota \mathcal{D}_\iota$. When $(\nu, \mu) \notin \mathcal{D}$, then Lemma 3.1 allows a definition of $\operatorname{sign}(\det(D_\iota)) \in \{\pm 1\}$ as in (2.13) for each $\iota \in H^1(C; \mathbb{Z}/2)$. (Thus, $\operatorname{sign}(\det(D_\iota))$ counts the mod (2) spectral flow for the path of D_ι operators along a generic path from (ν, μ) to any $(\nu_1, 0) \notin \cup_\iota \mathcal{D}_{0\iota}$.) As ι ranges over the four elements in $H^1(C; \mathbb{Z}/2)$, these four ± 1's give a map

$$\delta : H^1(C; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2. \tag{3.1}$$

Note that δ is not a homomorphism.

As remarked previously, an embedded, pseudo-holomorphic torus $C \subset X$ with trivial normal bundle has associated to it an operator D as in (2.7) and, more generally, the full set of four operators $\{D_\iota\}$. Thus, a map $\delta \equiv \delta_C$ as defined in (3.1) can be assigned to C when each $\{D_\iota\}$ has trivial kernel.

Step 2. With the preceding understood, the function $r(\cdot)$ will now be described.

Definition 3.2. Let $C \subset X$ be an embedded, pseudo-holomorphic torus with topologically trivial normal bundle and the property that each D_ι has kernel $\{0\}$. Define $r(\cdot)$ as follows:

1. If $\delta(\iota) = 1$ for all ι, set $r(C, m) = 1$.

2. If $\delta(0) = 1$ and $\delta(\iota) = -1$ for exactly one ι, set $r(C, 1) = 1$ and $r(C, m) = 0$ for $m > 1$.

3. If $\delta(0) = 1$ and $\delta(\iota) = -1$ for exactly two ι, set $r(C, 1) = 1$ and $r(C, 2k) = r(C, 2k + 1) = (-1)^k$ for $k \geq 1$.

4. If $\delta(0) = 1$ and $\delta(\iota) = -1$ for all other ι, then set $r(C, 1) = 1$ and $r(C, 2k) = r(C, 2k + 1) = (-1)^k 2$ for all $k \geq 1$.

5. If $\delta(0) = -1$ and $\delta(\iota) = 1$ for all other ι, set $r(C, 1) = -1$ and $r(C, m) = 0$ for all $m > 1$.

6. If $\delta(0) = -1$ and $\delta(\iota) = -1$ for exactly one other ι, set $r(C, 1) = -1$ and $r(C, m) = (-1)^m$ for all $m > 1$.

7. If $\delta(0) = -1$ and $\delta(\iota) = -1$ for exactly two other ι, set $r(C, 1) = -1$ and $r(C, m) = (-1)^m 2$ for all $m > 1$.

8. If $\delta(\iota) = -1$ for all ι, set $r(C, 2k) = -r(C, 2k+1) = 2k+1$ for all k.

Here are two remarks concerning this definition: First, the value of $r(C, 1)$ is equal to sign(det(D)) as required. Second, the value of $r(C, m)$ is, in general, the coefficient of z^m in the power series expansion around $z = 0$ for a certain analytic function $P(C; z)$ of the variable z. As $P(C; z)$ depends only on the value of the map δ_C, it proves useful to introduce the power series $P_{\pm k}$ where $P_{+k} = P(C; \cdot)$ in the case where the map δ for C sends the trivial element to 1 and otherwise sends precisely k elements to -1. Likewise, $P_{-k} = P(C; \cdot)$ in the case where the map δ for C sends the trivial element to -1 and precisely k additional elements to -1 also. With the preceding understood, the eight possibilities from Definition 3.2 are given by power series expansions (about 0) for the following:

$$
\begin{aligned}
&1.\quad P_{+0}(z) \equiv \frac{1}{1-z}.\\
&2.\quad P_{+1}(z) \equiv 1+z.\\
&3.\quad P_{+2}(z) \equiv \frac{1+z}{1+z^2}.\\
&4.\quad P_{+3}(z) \equiv \frac{(1+z)(1-z^2)}{(1+z^2)}.\\
&5.\quad P_{-0}(z) = 1-z.\\
&6.\quad P_{-1}(z) \equiv \frac{1}{1+z}.\\
&7.\quad P_{-2}(z) \equiv \frac{1+z^2}{1+z}.\\
&8.\quad P_{-3}(z) \equiv \frac{1+z^2}{(1+z)(1-z^2)}.
\end{aligned}
\tag{3.2}
$$

4. The meaning of the term "admissable"

The purpose of this step is to make a precise statement about the "admissable" set of almost complex structures on X and points Ω which can be used to define $\mathcal{H}$ for the sum in (1.7). For this purpose, it is necessary for a digression to introduce a somewhat different generalization of the operator D.

To start the digression, let C be, again, a compex torus, and let N be a topologically trivial, complex vector bundle over C. With a positive integer m fixed, let P_m denote the group of permutations of the set $\{1, \ldots, m\}$. Fix a homorphism $\rho : \pi_1(C) \longrightarrow P_m$. Via the obvious representation of P_n on $\oplus_n \mathbb{R}$, the representation ρ naturally defines a flat, real n-plane bundle $V_\rho \longrightarrow C$. With this understood, the operator D naturally extends to an $\mathbb{R}$-linear operator (also called D) on the space of sections of $V_\rho \otimes N$.

Definition 4.1. Let $C \subset X$ be a pseudo-holomorphic submanifold with genus one and topogically trivial normal bundle N. Fix a positive integer n and say that C is <u>n-non-degenerate</u> when, for all $m \in \{1, \ldots, n\}$, and for all representations $\rho : \pi_1(C) \longrightarrow P_m$, the operator D on the space of sections of $V_\rho \otimes N$ has trivial kernel. Equivalently: C is n-nondegenerate if, for every holomorphic covering map $f : C' \to C$ of degree n or less, the operator D' on C', given by $(2, 7)$ with f_N^*, f_ν^*, f_μ^* replacing N, ν, μ respectively, has trivial kernel.

Here are three remarks concerning this definition: First, the notion of 1-non-degeneracy is equivalent to that of non-degeneracy as given in Definition 2.1. Second, if C is 2-non-degenerate, then each of the operators D_ι has trivial kernel. Third, the equivalence of the two definitions will become apparent below.

End the digression.

To describe the set of admissable (J, Ω) for the sum in (1.7) it proves useful to introduce the space $\mathcal{A}_d$ of pairs (J, Ω), where J is an ω-compatible almost complex structures on TX, and Ω is a set of d distinct points in X when d is positive, and otherwise, $\Omega = \emptyset$. This $\mathcal{A}_d$ has the structure of a smooth manifold which it inherits as a subspace of the Frechet manifold $C^\infty(\mathrm{End}(TX)) \times \mathrm{Sym}^d(X)$.

Definition 4.2. A pair (J, Ω) of ω-compatible almost complex structure and set $\Omega \subset X$ of $m \geq 0$ distinct points will be called <u>admissable</u> when the five points below hold for each $e \in H^2(X; \mathbb{Z})$ for which the number d (as defined in (1.3)) is no greater than m:

1. There are but finitely many connected pseudo-holomorphic submanifolds with fundamental class Poincaré dual to e and containing d points of Ω.

2. Each of the submanifolds in Point 1, above, is non-degenerate in the sense of Definition 2.1.

3. There are no connected, pseudo-holomorphic submanifolds with fundamental class Poincaré dual to e and containing more than d points of Ω.

4. There is an open neighborhood of (J, Ω) in $\mathcal{A}_m$ with the property that each (J', Ω') in this neighborhood obeys Points 1-3, above; and the number of J'-pseudo-holomorphic submanifolds in Point 1, above, is constant as (J', Ω') vary through this neighborhood.

5. If $e \cdot e = e \cdot c = 0$, then each pseudo-holomorphic submanifold in Point 1 is n-non-degenerate for each positive integer n.

The following proposition asserts that admissable (J, Ω) are generic.

Proposition 4.3. *Fix a class $e \in H^2(X; \mathbb{Z})$ and introduce d as in (1.3). Then, the set of admissable pairs (J, Ω) in $\mathcal{A}_d$ is a Baire subset. Furthermore, if (J, Ω) is admissable, then the following hold:*

1. The set $\mathcal{H}(e, J, \Omega)$ as defined by J and Ω is a finite set.

2. $\mathcal{H}(e, J, \Omega = \emptyset)$ is empty when $d < 0$.

3. Every $h = \{(C_k, m_k)\} \in \mathcal{H}$ has the property that each C_k with $m_k = 1$ is non-degenerate in the sense of Definition 2.1, while each C_k with $m_k > 1$ is m_k-non-degenerate in the sense of Definition 4.1.

4. If (J_1, Ω_1) is sufficiently close to (J, Ω), then the sets $\mathcal{H}(e, \Omega, J)$ and $\mathcal{H}(e, \Omega_1, J_1)$ have the same number of elements.

(A Baire set is the countable intersection of open and dense sets. In particular, such a set is dense.)

This proposition is proved in Section 5.

5. The proofs

The purpose of this section is to provide the proofs of Lemma 3.1, Proposition 4.3 and Theorem 1.1. These are taken in the preceding order.

a) Proof of Lemma 3.1.

The proof of this lemma is a six step affair.

Step 1. The purpose of this first step is to give a very concrete realization of the operator D in (2.7) for the case, where C is a compex torus, and N is topologically trivial. Before starting, it is convenient to redefine the function ν. The point is that by adding to or subtracting from ν and by making the complimentary change in the definition of the operator $\overline{\partial}$, one can change the implicit holomorphic structure on $\epsilon_\iota \otimes N$ to be the trivial one. With this change understood, one should think of C as $\mathbb{C}/H$ where $H \subset \mathbb{C}$ is a lattice generated by 1 and τ which has positive imaginary part. Use $z \equiv t_1 + i \cdot t_2$ for the coordinate on $\mathbb{C}$ so that $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)$ in (2.7). Now, the complex valued 1-form $dt_1 + i \cdot dt_2$ defines a holomorphic trivialization of $T^{1,0}C$.

With $\epsilon_\iota \otimes N$ and $T^{1,0}C$ holomorphically triviallized, ν and μ in (2.7) become complex valued functions on C. And, the operator D_ι in (2.7) can be viewed as an $\mathbb{R}$-linear, differential operator on the space of complex valued functions on C. This very concrete view of D_ι will be invoked on occasion, below.

Note, by the way, that this device of redefining ν puts all of the D_ι in a similar form, and with this understood, the subsequent discussion will drop the subscript ι and use D to denote any of the four possibilities.

Step 2. The operator D in the $\mu \equiv 0$ case defines a $\overline{\partial}$ operator on the topologically trivial complex bundle over C. The kernel of D in this case consists of the holomorphic sections. Since the bundle is topologically trivial, a holomorphic section is non-vanishing. Thus, there is, at most, a complex 1-dimensional subspace of holomorphic sections.

Now, suppose that $D\eta = 0$ with $\eta \neq 0$. Since η is nowhere zero, η defines a smooth map from C to $\mathbb{C} - \{0\}$. Conversely, take a smooth map $\eta : C \longrightarrow \mathbb{C} - \{0\}$, set $\nu \equiv -\eta^{-1}\overline{\partial}\eta$. With this choice for ν to use in D, one has $D\eta = 0$. This last construction identifies $\mathcal{D}_0$ with the space Maps $(C; \mathbb{C} - \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of non-zero, complex numbers (i.e., the constant maps from C to $\mathbb{C} - \{0\}$). With the preceding understood, the components of $\mathcal{D}_0$ are seen to be in 1-1 correspondence with the homotopy classes of maps from C to $\mathbb{C} - \{0\}$. (The latter is classified by $H_1(C; \mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$).

Now, Hodge theory asserts, in general, that ν can be written

$$\nu = \alpha + \overline{\partial}u, \tag{5.1}$$

where α is a constant, and u is a smooth function on C. One thus sees that (5.1) has the form $-\nu^{-1}\bar{\partial}\nu$ if and only if

$$\alpha = i\pi \cdot (n_1 + \frac{i}{\operatorname{Im}(\tau)}(-n_1 \cdot \operatorname{Re}(\tau) + n_2)), \tag{5.2}$$

where n_1 and n_2 are integers. This last observation establishes the first assertion of the lemma.

Step 3. This step proves the following lemma which is of general use later on.

Lemma 5.1. *Let C be a complex torus and let (ν, μ) be a pair of complex valued functions on C. Use (2.7) to define the $\mathbb{R}$-linear operator D on the space of complex valued functions on C. If η obeys $D\eta = 0$, then η is nowhere zero.*

Proof of Lemma 5.1. Aronszajn's unique continuation theorem [1] asserts that η vanishes to finite order at any one of its zeros. Then, a Taylor's expansion near a putative zero shows that η vanishes holomorphically. That is, $\eta = a \cdot z^p + \mathcal{O}(|z|^{p+1})$ for some integer p, non-zero constant a and local holomorphic coordinate z. This last fact implies that the zeros of η count only with positive signs in a computation of the Euler characteristic of the trivial bundle. Hence, η is nowhere zero.

Step 4. The observation here is that the claim in Assertion 2 about the dimension of the kernel of D follows directly from Lemma 5.1.

Step 5. This step considers the structure of $\mathcal{D} \equiv \mathcal{D}_\iota$. To begin, suppose that kernel$(D) \neq \{0\}$ for a given pair (ν, μ). Analytic perturbation theory can be used to describe $\mathcal{D}$ in a neighborhood of (ν, μ) as follows: There is a ball, $\mathcal{B}$, about (ν, μ) in $\times_2 C^\infty(C; \mathbb{C})$ and a real analytic map F from $\mathcal{B}$ into Hom(kernel(D), cokernel(D)) with the property that $F^{-1}(0) = \mathcal{D} \cap \mathcal{B}$. Thus, $\mathcal{D}$ is a real analytic variety near (ν, μ).

To determine the codimension of $\mathcal{D}$, consider that the differential of the map F at (ν, μ) sends a pair (ν_1, μ_1) to the homomorphism which takes $\eta \in$ kernel(D) and gives the L^2- orthogonal projection onto cokernel(D) of $\nu_1 \cdot \eta + \mu_1 \cdot \bar{\eta}$. The claim is that this differential is not zero. Indeed, take non-zero $\eta \in$ kernel(D) and $\theta \in$ cokernel(D) to make $(\nu_1 \equiv \bar{\theta}\eta, \mu_1 \equiv 0)$ which is not annihilated by the differential of F.

The fact that F's differential is not zero implies the assertion that the strata of $\mathcal{D}_\iota$ all have codimension 1 or more. (The preceding implies that where kernel(D) has dimension-1, the variety $\mathcal{D}_\iota$ is a codimension-1 submanifold.)

The assertion about the structure of $\mathcal{D}_\iota \cap \mathcal{D}'_\iota$ follows from the general form of F above. The details are left to the reader.

Step 6. This step proves Part 3 of Lemma 3.1. To begin, remark that the association of non-zero $s \in \text{kernel}(D)$ to the homotopy class of s (as a map from C into $\mathbb{C} - \{0\}$) defines a locally constant map from $\mathcal{D}$ into $\mathbb{Z} \times \mathbb{Z}$. (Note that where D has 2-dimensional kernel, any two elements of said kernel are either proportional or everywhere linearly independent. In particular, they define the same homotopy class of map in to $\mathbb{C} - \{0\}$.) Conversely, given a complex valued function μ on C and a smooth map s from C into $\mathbb{C} - \{0\}$, set $\nu \equiv -s^{-1}\overline{\partial}s - s^{-1}\mu\overline{s}$ and then define D using (ν, μ). This D annihilates s. Thus, the afore-mentioned locally constant map to $\mathbb{Z} \times \mathbb{Z}$ is onto.

With the help of the preceding construction, one can also see that each point in $\mathcal{D}$ is connected to a point in $\mathcal{D}_0$ by a path in $\mathcal{D}$. For example, the path for (ν, μ) as above is parameterized by $r \in [0, 1]$ where $\mu_r \equiv r \cdot \mu$ and $\nu_r \equiv -s^{-1}\overline{\partial}s - r \cdot s^{-1}\mu\overline{s}$. (Note that this does not define a retract of $\mathcal{D}$ onto $\text{Maps}(C; \mathbb{C} - \{0\})/\mathbb{R}^*$ because a given D may have a 2-dimensional kernel.) At any rate, the existence of paths as above demonstrates that the components of $\mathcal{D}$ are, as claimed, in 1-1 correspondence with the set of pairs of integers.

b) Proof of Proposition 4.3, Part 1.

The proof of Proposition 4.3 (and Theorem 1.1) depends critically on results from Section 5 of [7], which describe the local structure, and also the compactness properties of certain relevant spaces of connected, pseudo-holomorphic submanifolds. Part 1 of the proof of Proposition 4.3 summarizes these results as Proposition 5.2, below. Part 2 completes the argument.

As remarked, the purpose of this step is to state and prove Proposition 5.2, below. The statement of Proposition 5.2 requires the following four part digression.

For Part 1 of this digression, suppose that Σ is a compact, oriented, connected, 2-dimensional surface. Fix $e \in H^2(X; \mathbb{Z})$. Let ω be a symplectic form on X and let J be an ω-compatible, almost complex structure on X. Also, introduce d as in (1.3), and if $d \geq 0$, let Ω be a set of d distinct, but unlabeled points in X. Introduce $\mathcal{K}(J, \Omega)$ to denote the set of pseudo-holomorphic submanifolds in X which are (abstractly) diffeomorphic to Σ, contain the set Ω, and have fundamental class that is Poincaré dual to e. Note that this set is empty unless the genus g of Σ is given by the adjunction formula (1.2). End Part 1 of the digression.

Part 2 of the digression introduces the notion of a multiply toroidal class $e \in H^2(X;\mathbb{Z})$. A class e is multiply toroidal when all of the following three conditions hold:

$$
\begin{array}{ll}
1. & e \cdot e = 0. \\
2. & e \cdot c = 0. \\
3. & e \text{ is divisible.}
\end{array} \tag{5.3}
$$

End Part 2 of the digression.

For Part 3 of the digression, let $\mathcal{A}$ denote the Frechet manifold of ω-compatible, almost complex structures on X. (This is a submanifold in the space of smooth sections of $\mathrm{End}(TX)$.) When d is positive, let X_d denote the space of d-tuples of distinct (but unlabeled) points in X (a smooth manifold). When $d \leq 0$, set $X_d = \emptyset$. In either case, set $\mathcal{A}_d = \mathcal{A} \times X_d$.

For Part 4 of the digression, consider a smooth, 1-parameter family, $\{\omega_t : t \in [0,1]\}$, of symplectic forms on X. Let $\underline{\mathcal{A}_d}$ denote the set of triples (t, J, ω), where $t \in [0,1]$, J is an ω_t-compatible symplectic form on X, and Ω is as before. Note that $\underline{\mathcal{A}_d}$ is a smooth, Frechet manifold which fibers over $[0,1]$ via the tautological projection.

End the digression.

Proposition 5.2. *Fix a class $e \in H^2(X;\mathbb{Z})$ and, if $e \cdot e = 0 = c \cdot e$, choose a positive integer n. There is an open and dense set $\mathcal{U} \subset \mathcal{A}_d$ with the following property: When (J, Ω) is chosen from $\mathcal{U}$, then*

1. *$\mathcal{K}(J, \Omega)$ is a finite collection of points, and each point in question is non-degenerate in the sense of Definition 2.1. Furthermore, if $e \cdot e = 0 = c \cdot e$, then each point in $\mathcal{K}(J, \Omega)$ is n-non- degenerate in the sense of Definition 4.1.*

2. *There is an open neighborhood of (J, Ω) in $\mathcal{A}_m$ with the property that each (J', Ω') in this neighborhood obeys Assertion 1, and its $\mathcal{K}(J', \Omega')$ has the same number of points as $\mathcal{K}(J, \Omega)$.*

3. *Suppose that $\{\omega_t : t \in [0,1]\}$ is a smooth, 1-parameter family of symplectic forms. Then, a point in $\mathcal{U}$ (as defined by ω_0) can be joined by a section $\gamma : [0,1] \longrightarrow \underline{\mathcal{A}_d}$ to a point in $\mathcal{U}$ (as defined by ω_1) with the property that the fibered product,*

$$
\mathcal{X}_\gamma \equiv \{(t, \xi) : t \in [0,1] \text{ and } \xi \in \mathcal{K}(\gamma(t))\} \tag{5.4}
$$

has the structure of an oriented, 1-manifold. Furthermore, this manifold is compact when e is not multiply toroidal.

Proof of Proposition 5.2. This proposition is essentially proved in Section 5 of [7]. (As remarked in the Introduction, there is an oversight in Section 6 of [7], but this does not effect the proof of Proposition 5.2.)

Briefly, the proof (borrowed mostly from [7]) of Proposition 5.2 proceeds by setting up a unia universal model for $\mathcal{K}(\cdot)$ and certain analogous spaces of pseudo-holomorphic varieties. The argument then exploits the Fredholm properties of D in conjunction with the Sard-Smale theorem [9] and the Gromov compactness theorem (see [2], [6] and [15] and [5]) to rule out unwanted behavior for generic pairs (J, Ω). What follows is a brief eight step description of the universal model and its application. There is also an addendum at the end.

Step 1. Fix a class $e \in H^2(X; \mathbb{Z})$. Let Σ be any compact, oriented, connected surface. Introduce the space $\mathcal{M}$ of smooth maps from Σ to X which push-forward the fundamental class of Σ as the Poincaré dual of e and which are embeddings off of a finite set of points. Note that the group $\mathrm{Diff}(\Sigma)$ of orientation preserving diffeomorphisms of Σ acts freely on $\mathcal{M}$.

Step 2. Let $\mathcal{J}$ denote the space of almost complex structures on Σ (compatible with its orientation). Think of $\mathcal{J}$ as a Frechet submanifold of the space of smooth endomorphisms of $T\Sigma$. The group $\mathrm{Diff}(\Sigma)$ also acts on $\mathcal{J}$ and the quotient is the moduli space of complex structures on Σ. The latter has a natural structure of a complex analytic variety. In particular, $\mathcal{J}/\mathrm{Diff}(\Sigma)$ is stratified; that is, it is the disjoint union of complex analytic manifolds,

$$\mathcal{J}/\mathrm{Diff}(\Sigma) = L_0 \cup L_1 \cup \dots, \tag{5.5}$$

where L_k is a smooth manifold of dimension $6g-6+\epsilon-2k$, where g is the genus of Σ and where $\epsilon = 6, 2$ or 0 depending on whether $g = 0$, $g = 1$, or $g > 1$. Use $\mathcal{J}_k \subset \mathcal{J}$ to denote the subspace which projects to L_k. (The membership of a complex structure in a component of a strata of (5.5) is determined by the conjugacy class of the group of holomorphic diffeomorphisms. For example, if $g > 0$, the generic complex structure admits only the identity diffeomorphism in this group; and if $g = 0$, the generic complex structure admits only the torus of translations.)

Step 3. A universal model consists of the space of $\mathrm{Diff}(\Sigma)$ orbits of data $([(z), (y), j, \varphi], J, (x))$ where $(z) \equiv (z_1, \dots, z_m)$ and $(y) =$

$(y_1, \dots, y_p)$ are sets of unordered, distinct points in Σ. Meanwhile, $(x) = (x_1, \dots, x_m)$ is a set of unordered, distinct points in X. Also, $j \in \mathcal{J}_k$ for some k, while $\varphi \in \mathcal{M}$ and $J \in \mathcal{A}_0$. This data set is constrained by the following requirements:

(5.6)
1. $\varphi_* j = J \varphi_*$.
2. The differential of φ vanishes at each y_k.
3. $\varphi(z_k) = x_k$.
4. φ is an embedding off of a finite set of points.

Step 4. Using the fact that D in (2.7) is elliptic, the arguments in Chapters 3.2 and 6.1 in [5] can be generalized in a rather straightforward manner to prove that the space of $\mathrm{Diff}(\Sigma)$ orbits of such data sets is a smooth manifold for which the tautological map to $\mathcal{A}_m$ has everywhere Fredholm differential.

The following paragraphs constitute a brief outline of the argument: The discussion starts with new notation. Introduce the "universal space" $\mathcal{Z} = (\Sigma^m \times \Sigma^p \times \mathcal{J}_k \times \mathcal{M})/\mathrm{Diff}(\Sigma) \times \mathcal{A}_m$. Next, consider three fiber bundles over $\mathcal{Z}$. The first is the vector bundle $\mathcal{W}$; its fiber depends only on the triple (j, φ, J) and is the space of sections over Σ of the bundle $T^{0,1}\Sigma \otimes \varphi^*(T_{1,0}X)$. Here, $T^{0,1}\Sigma$ is defined using j, and $T_{0,1}X$ is defined using J. The second bundle is the vector bundle V_p; in this case, the fiber depends only on the data $((y), j, \varphi, J)$; it is $\oplus_p[(T^{1,0}\Sigma \otimes (\varphi^*(T_{1,0}X))|_{y_p}]$. The third fiber bundle is the trivial fiber bundle $W_m = X_m \times \mathcal{Z}$.

With the proceding understood, consider that lines 1-3 of (5.6) can be interpreted as defining a section, s, over z of the fiber bundle $(\mathcal{W} \oplus V_p) \oplus W_m$. And, the space of solutions to the constraints in (5.6) constitute the set of points in $\mathcal{Z}$ for which the section s intersects a certain tautological section s_0. The latter is the section which sends $([(z), (y), j, \varphi], J, (x))$ to $(0, 0, (x))$.

The claim now is that the differential of s is surjective at points where $s = s_0$. The proof of this claim is left as an exercise (see Sections 3.2 and 6.1 of [5]) except for the following explanatory remarks. First, to be technically precise, one should choose some Sobolev space or Hölder space completions of $\mathcal{Z}$, that is, of $\mathcal{M}$ and $\mathcal{A}_0$ and $\mathcal{J}_k$, and so work with infinite dimensional Banach manifolds. This formality is explained in the aforementioned sections of [5], and will receive no further mention save for one further technical remark below. Second, one should consider $\mathcal{Z}$ as a fiber bundle over the manifold L_k. When this is done, the arguments are similar to those in Chapters 3.2 and 6.1 of [5] which treat the genus 0 case where the corresponding L_k is a point. Indeed, as in [5], one finds that the differential of s along $\mathcal{M}$, thought of as a factor in the fiber of this bundle over L_k, is surjective except for what is, at most, a finite dimensional vector space. This is because the equations for a map to be pseudo-holomorphic have elliptic and hence Fredholm linearization. Then, variations along $\mathcal{A}_0$ and X_m map onto this last finite dimensional piece of the range. In fact, variations along L_k are not needed when considering the surjectivity of this differential; and this makes the argument even closer to that in [5].

Given that the set $\mathcal{Z}_0 = \{s = s_0\} \subset \mathcal{Z}$ is a smooth manifold, one can then consider the induced map from $\mathcal{Z}_0$ to $\mathcal{A}_m$. It is a consequence of the Sard-Smale theorem [9] (see the technical remark at the end of this step) that the regular values of this map to $\mathcal{A}_m$ form a Baire set. (Remember that Baire sets are dense.) Meanwhile, the inverse image of a regular value is a smooth manifold whose dimension is equal to the index of the differential of the map to $\mathcal{A}_m$. That the map to $\mathcal{A}_m$ is Fredholm follows as a direct consequence of the fact that the linearization of the condition for pseudo-holomorphicity is Fredholm.

These last points imply that $\mathcal{K}(J, \emptyset)$ is empty for generic J when $d < 0$, and that $\mathcal{K}(J, \Omega)$ is a smooth 0-dimensional manifold for generic (J, Ω) when $d \geq 0$. Here, the term generic means chosen from a Baire set of regular values of a map whose differential is everywhere Fredholm with index -2 or less. These arguments use the $p = 0$ and $\mathcal{M} = \mathcal{D}$ versions of (5.6).

Note that the definition of the term regular value yields the following: When (J, Ω) is a regular value for the map to $\mathcal{A}_m$, then each point in $\mathcal{K}(J, \Omega)$ parametrizes a submanifold which is non- degenerate in the sense of Definition 2.1.

Similar arguments with the $m > d$ versions of (5.6) find a Baire set (the regular values of a map whose differential is Fredholm with index ≤ -2) of pairs $(J, \Omega_1) \in \mathcal{A}_m$ with no pseudo-holomorphic maps from Σ into X, which push-forward $[\Sigma]$ as the Poincaré dual to e and hit all the points in Ω_1.

Likewise, arguments with the $p \neq 0$ versions of (5.6) find a similar Baire set of (J, Ω) where no complex structures on Σ have pseudo-holomorphic maps into X, which push the fundamental class forward as the Poincaré dual to e, hit all points in Ω, and embed the compliment of a finite set, yet do not embed all of Σ.

Here is a technical remark which concerns the preceding discussions:

Recourse to the Sard-Smale theorem for maps between the Frechet manifolds used here requires a detour to introduce a certain countable, nested sequence of separable Hilbert manifold thickenings of the spaces involved. These Hilbert manifolds are modeled on Sobolev spaces of functions whose derivatives to some fixed order are square integrable. The order here increases without bound, and gives the indexing for the sequence. The asserted existence of a Baire set for the smooth model (that is, $\mathcal{A}_d$) follows from the existence of an analogous open and dense set for a certain subset in each of the Hilbert manifold thickenings. (Note that each $\mathcal{K}(J,\Omega)$ from the Sobolev space version of z_0 is locally compact and thus can be exhausted by compact sets. That is, the projection from $\mathcal{Z}_0$ to $\mathcal{A}_m$ in the Sobolev space version is a proper map. Note also that the countable intersection of Baire sets is still Baire.) The use of a sequence of Sobolev space thickenings of $\mathcal{Z}_0$ to find a Baire set in the smooth version of $\mathcal{A}_m$ of regular values is described in Chapter 3.2 of [5]. Similar arguments which use the Sard-Smale theorem on a countable set of Sobolev space versions of $\mathcal{Z}_0$ are used below to find other Baire subsets of the smooth version of $\mathcal{A}_m$. In these subsequent discussions, the technical detour through a sequence of Sobolev space thickenings will be left implicit.

Step 5. This step considers sequences $\{(C_m,(J_m,\Omega_m))\}$, where $\{(J_m,\Omega_m)\}$ converges to (J,Ω), and C_m for each m is a point in $\mathcal{K}(J_m,\Omega_m)$. The goal here is to prove that when (J,Ω) is chosen from the appropriate Baire set, then there must be a subsequence of $\{C_m\}$ which converges to a J-pseudo-holomorphic submanifold C whose fundamental class is Poincaré dual to e. With this last assertion proved, then Assertions 1 and 2 of Proposition 5.2 follows by a straightforward argument. (This argument invokes a fairly standard application of the implicit function theorem to prove the following: If a point $C \in \mathcal{K}(J,\Omega)$ is non-degenerate (as in Definition 2.1), then whenever (J_1,Ω_1) is sufficently close to (J,Ω), there is a unique $C_1 \in \mathcal{K}(J_1,\Omega_1)$ which is close to C. (And, C_1 is also non-degenerate.))

With the preceding understood, then the issue is the behavior of a sequence $\{C_m\}$ of J_m-pseudo-holomorphic submanifolds, which has no subsequence that converges to a submanifold. The principle tool for studying the properties of such a sequence is the Gromov compactness theorem [2] as described [6] and [15] (see also [5]). This compactness theorem details the sorts of limits which can arise. It turns out that each of these limits can be described by some version or other of (5.6). And, except when e is multiply toroidal, all of the versions of (5.6) which arise via the Gromov compactness theorem have the property that the index of the relevant map to the relevant $\mathcal{A}_*$ is at most -2. This implies that there is a Baire set of choices for (J, Ω) for which no possible Gromov limit exists. For such (J, Ω), all sequences $\{(C_m, (J_m, \Omega_m)\}$, as described above, have convergent subsequences; and Assertions 1 and 2 of Proposition 5.2 follow as described, at least when e is a non-multiply toroidal classes.

In the case where e is multiply toroidal, the argument for Assertions 1 and 2 is more complicated; see the next step of the proof.

Here is a four part argument to prove the assertion above about the index for the universal models which come via the Gromov compactness theorem.

Part 1. From a sequence $\{(C_m, (J_m, \Omega_m)\}$ with $\{(J_m, \Omega_m)\}$ converging to (J, Ω), the Gromov compactness theorem gives a finite set of data $\{(\varphi_k, \Sigma_k, m_k)\}$, where Σ_k is a connected, complex curve, φ_k is a J-pseudo-holomorpic map from Σ_k into X which is an embedding off of a finite set of points, and where m_k is a positive integer. This data is further constrained: First, $\Sigma_{k_1} \cap \Sigma_{k_2}$ is finite if $k_1 \neq k_2$. Second, let e_k denote the Poincaré dual to the φ_k-push forward of $[\Sigma_k]$. Then $e = \Sigma_k m_k e_k$. Third, $\cup_k \varphi_k(\Sigma_k)$ is connected and contains Ω. Fourth, $[\omega] \cdot e_k > 0$ with $[\omega]$ the class of ω.

Part 2. Now, introduce the integer d for e as given in (1.3), and introduce the analogous integer d_k for each e_k. The first point to make is that the arguments of Step 4, above, can be repeated to show that when (J, Ω) is chosen from an appropriate Baire set, then each d_k can be assumed non-negative. Furthermore, one can choose the Baire set to insure that the inequality $\Sigma_k\, d_k \geq d$ is satisfied. (The d points of Ω have to be contained in the φ_k-images of the curves Σ_k, and the arguments from Step 4 show that if (J, Ω) is suitably generic, each point of Ω is in the image of unique φ_k, and furthermore, $\varphi_k(\Sigma_k)$ can not contain more than d_k such points.) In both these cases, the Baire sets in question consists of the regular values of a map whose linearization is everywhere Fredholm with index -2 or less.

Part 3. This part asserts that if (J, Ω) is chosen from the appropriate Baire set, then $\Sigma_k\, d_k < d$, unless either

1. $\{C_m\}$ has a subsequence which converges to a J-pseudo-holmorphic submanifold C with fundamental class Poincaré dual to e; or
2. e is multiply toroidal and the data given by the Gromov compactness theorem consists solely of one triple $(\varphi_1, \Sigma_1, m_1)$. In addition,

Σ_1 is a torus, φ_1 embeds $\Sigma_1, m_1 > 1$ and $e = m_1 e_1$.

(This Baire set consists, again, of the regular values of a map whose differential is everywhere Fredholm with index -2 or less.)

Part 4. This last part proves the assertion in Part 3. For this purpose, it is useful to label the set $\{e_k\}$ so that the classes with negative square are given as $\{e_1, \dots, e_p\}$. By arguments such as those in Step 4, one can prove that when (J, Ω) is chosen from a certain Baire set (as will be assumed from now on), then each e_k for $k \leq p$ has square -1, the corresponding Σ_k is a 2-sphere, and the corresponding φ_k is an embedding whose image is regular in the sense of Definition 2.1. Since distinct pseudo-holomorphic submanifolds intersect with only locally positive intersection number [4], it follows that the set $\{e_1, \dots, e_p\}$ consists of distinct classes. Furthermore, either each has non-negative cup product pairing with e, or $p = 1$ and $e = e_1$. This is because for all m sufficiently large, each such class is Poincaré dual to the fundamental class of a J_m-pseudo-holomorphic submanifold, and so is e, by assumption. If $e = e_1$, the argument is finished, so assume below that this is not the case.

With e given as $\Sigma_k m_k e_k$, compute the integer d as in (1.3) in terms of the analogous integers d_k for the classes e_k with non-negative square. (That is, with $k > p$.) Here is the result:

$$
\begin{aligned}
d = &\Sigma_k m_k d_k + \frac{1}{2}\Sigma_{k>p} \ m_k(m_k - 1)(e_k \cdot e_k) \\
&+ \Sigma_{k_1>k_2>p} \ m_{k_1} m_{k_2}(e_{k_1} \cdot e_{k_2}) \\
&+ \Sigma_{k_1>p, k_2 \leq p} \ m_{k_1} m_{k_2}(e_{k_1} \cdot e_{k_2}) \\
&+ \Sigma_{k_1<k_2 \leq p} \ m_{k_1} m_{k_2}(e_{k_1} \cdot e_{k_2}) \\
&- \frac{1}{2}\Sigma_{k \leq p} \ m_k(m_{k-1}).
\end{aligned} \tag{5.7}
$$

Notice that the first three terms are non-negative when (J, Ω) comes from the appropriate Baire set. Indeed, the Baire set condition insures that all $d_k \geq 0$ (by arguments such as those from Step 4). Meanwhile, the second term is non-negative by assumption, and the third term is non-negative because of the local positivity of the intersection of pseudo-holomorphic curves (see [4]). Infact, the only term which is not evidently non-negative is the final term. However, because e_k for $k \leq p$ has non-negative intersection with e, it follows that the sum of the final three terms in (5.7) is no smaller than $\frac{1}{2}\Sigma_{k \leq p} m_k(m_{k+1})$.

This implies that d is strictly larger than $\Sigma_k\, d_k$ unless the following are true:

1. All e_k have non-negative square; that is $p = 0$.
2. $m_k = 1$ unless $d_k = 0$ and $e_k \cdot e_k = 0$.
3. If $k_1 > k_2$, then $e_{k_1} \cdot e_{k_2} = 0$.

Since $\cup\, \varphi_k(\Sigma_k)$ is connected, and pseudo-holomorphic curves intersect locally positively, this last conclusion implies that either $d > \Sigma_{k>p}\, d_k$ or else one of the two possibilities claimed in Part 3 holds. (That is, there is just an e_1; and $e_1 = e$ unless e is multiply toroidal, in which case e is a multiple of e_1.)

Step 6. The discussion in Step 5 proves Assertions 1 and 2 of Proposition 5.2 when e is not multiply toroidal. Now consider the case where e is multiply toroidal. Here, the preceding arguments and a careful analysis of the Gromov compactness theorem lead to Lemma 5.3, below. The statement of the lemma requires a brief, preliminary digression. To start the digression, reintroduce the flat vector bundle V_ρ over C as described in Section 3. Here, ρ is a representation of $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$ onto the permutation group of the set $\{1, \ldots, n\}$ for some n. For the purposes of the lemma below, remark that $\pi_1(C)$ acts naturally on V_ρ by permuting the $\mathbb{R}$ factors according to the representation ρ. (In this regard, remember that ρ is abelian.)

As remarked in Section 3, the operator D extends naturally to an $\mathbb{R}$-linear operator on the space of sections of $V_\rho \otimes \mathbb{C}$. Note that D commutes with the action via ρ of $\mathbb{Z} \oplus \mathbb{Z}$ on $V_\rho \otimes \mathbb{C}$, and thus there is an action of $\mathbb{Z} \oplus \mathbb{Z}$ on the kernel and also the cokernel of D. With this understood, when $s \in \text{kernel}(D)$, introduce $O_s \subset \text{kernel}(D)$ to denote the subspace which is spanned by the s and its translates under the $\mathbb{Z} \oplus \mathbb{Z}$ action. Similarly, define O_s when $s \in \text{cokernel}(D)$.

Lemma 5.3. *Suppose $e \in H^2(X;\mathbb{Z})$ is a multiply toroidal class. There is a dense, open set $\mathcal{U}_1 \subset \mathcal{A}_0$ with the property that when J comes from $\mathcal{U}_1$, then the following hold:*

1. *Each point in $\mathcal{K}(J, \emptyset)$ is non-degenerate in the sense of Definition 2.1.*

2. *If $\mathcal{K}(J, \emptyset)$ is not a finite set, or, if there exist pairs (ω_1, J_1) of symplectic form ω_1 and ω_1-compatible complex structure J_1 arbitrarily close to (ω, J) with a different size $\mathcal{K}(\cdot)$, then the following is true: First, write $e = n \cdot e_1$ for some integer $n > 1$. Then, there is a pair of positive integers (q, p) with $q < n$ and $p > 1$ which*

are such that qp divides n; and there is an embedded, J-pseudo-holomorphic torus whose fundamental class is Poincaré dual to $q \cdot e_1$ and which is not p-non-degenerate in the sense of Definition 4.1.

3. *To be more precise, under the preceding assumptions, there exists a representation ρ of $\mathbb{Z} \oplus \mathbb{Z}$ into P_p which acts transitively on $\{1, \dots, p\}$; and there is a non-zero section s of the corresponding $V_\rho \otimes \mathbb{C}$ which is annihilated by D. Furthermore, when $p > 2$, then the subspace O_s has dimension at least 2. And, when $p = 2$, then s is not a fixed point of the $\mathbb{Z} \oplus \mathbb{Z}$ action on* kernel(D).

Thus, the multiply toroidal case of Proposition 5.2 follows from Lemma 5.3 plus

Lemma 5.4. *Let X be a compact, oriented, 4-manifold with symplectic form ω. Let $e \in H^2(X; \mathbb{Z})$ obey the first two conditions in (5.3), and let n be a positive integer. There is an open and dense subset of smooth, ω-compatible almost complex structures J on X with the following property: Every embedded, pseudo-holomorphic torus whose fundamental class is Poincaré dual to e is n-non-degenerate in the sense of Definition 4.1.*

These two lemmas are proved in Step 8, below.

Step 7. The final statement of Proposition 5.2 is also a consequence of the following:

1. The Smale-Sard theorem.

2. In the case where e is not multiply toroidal, the upper bound of -2 for the index of the differential of the tautological maps to $\mathcal{A}_m$ from all auxilliary universal models given by the Gromov compactness theorem.

One version of the argument proceeds as follows: Generalize the universal model in (5.6) to consist of the Diff(Σ) orbits of points $([(z), (y), j, \varphi], (t, J, (x)))$ where $(t, J, (x)) \in \underline{\mathcal{A}_m}$. This generalized model (call it $\mathcal{Y}$) is also a smooth manifold for which the tautological map to $\underline{\mathcal{A}_m}$ has everywhere Fredholm differential.

Let $\mathcal{P}$ denote the Frechet space of smooth sections over $\underline{\mathcal{A}_m}$ with the property that at $t = 0$ and 1, the section reproduces the given points in the relevant versions of $\mathcal{U}$. Then, evaluation at time t gives a smooth map $ev : [0,1] \times \mathcal{P} \longrightarrow \underline{\mathcal{A}_m}$ which one can prove is transversal to the tautological map from $\mathcal{Y}$.

Thus, the fibered product over $\underline{\mathcal{A}_m}$ of $[0,1] \times \mathcal{P}$ (using ev) and $\mathcal{Y}$ (using the tautological map) is a smooth manifold whose tautological map to $\mathcal{P}$ has everywhere Fredholm linearization. (The fibered product sits in $[0,1] \times \mathcal{P} \times \mathcal{Y}$ as the set of $(t,p) \in [0,1] \times \mathcal{P}$ and $y \in \mathcal{Y}$ which are mapped (by ev and the tautological map) to the same point in $\underline{\mathcal{A}_m}$.) One now invokes the Sard-Smale theorem for this map from the fibered product to $\mathcal{P}$ to obtain Assertion 3. The compactness part of this assertion, for the case where e is not multiply toroidal, follows with the help of the Gromov compactness theorem in the manner described in Step 5 above. All non-compactness is described by auxilliary models for which the resulting map from the fibered product to $\mathcal{P}$ has differential with index -1.

Step 8. This step proves Lemmas 5.3 and 5.4.

Proof of Lemma 5.3. If the assumptions of Assertion 2 hold, then one can conclude from the discussion in Section 5 of [7] (or repeat, with minor changes, Step 5) that there exists a sequence $\{(\omega_m, J_m)\}$ of symplectic forms ω_m and ω_m-compatible, almost complex structures J_m which converge to (ω, J), and there is a corresponding sequence pairs (C_m, ψ_m), where C_m is a J_m-complex torus, and ψ_m is a J_m-pseudo-holomorphic embedding of C_m into X, which pushes forward the fundamental class of C_m as the Poincaré dual of e. Assume the existence of such a sequence.

The analysis [6] or [15] with a virtual repeat of arguments from Step 5 finds a certain Baire subset of $\mathcal{A}_0$, which consists of the regular values of a map whose linearization is a Fredholm operator with index-2 or less, with the property that when J comes from this set, then the following hold:

1. There is a positive integer $q < n$, a complex torus C and a J-holomorphic embedding, ψ, of C into X which pushes the fundamental class of the torus forward as the Poincaré dual of $q \cdot e_1$. Furthermore, if (J, Ω) are chosen from the appropiate open, dense set, then $\psi(C)$ can be assumed non-degenerate in the sense of Definition 2.1.

2. The sequence $\{C_m, \psi_m\}$ converges to a pair of complex torus C' and pseudo-holomorphic map $\psi' : C' \longrightarrow X$, where ψ' factors as $\psi \circ f$ with $f : C' \longrightarrow C$ being an n/q-sheeted, holomorphic

covering map.

Note that each C_m is diffeomorphic to C', only the complex structure changes with m. With this understood, the convergence here for $\{\psi_m\}$ is in the C^∞-topology on the space of smooth maps from the torus C' into X.

Now, identify C with its image in X, and introduce the disk bundle U in the normal bundle of C, with its embedding into X as described in (2.3). For large m, each ψ_m must map into the image of the disk bundle U via the map φ of (2.3), and thus can be considered as a map into U. By dilating each ψ_m by an appropriate m-dependent factor, one defines a second sequence of maps, $\{\psi_{1m} : C_m \longrightarrow N_C\}$ with the property that the farthest point from C in the image of the dilated C_m has distance 1 from C. The compactness theorem from Gromov [2], [6], [15], can then be invoked to prove that the sequence $\{\psi_{1m}\}$ has an infinite subsequence which converges to give a pseudo-holomorphic map $\psi_0 : C' \longrightarrow N_C$. Here, pseudo-holomorphic is defined with respect to the almost complex structure in (2.5), where $k \equiv 0$, and h is given by the first two terms only on the right side of (2.6).

Here are some useful observations concerning ψ_0: First, the image of ψ_0 does not intersect the zero section. Indeed, ψ_0 cannot factor through the zero section as there is at least one point in its image with distance 1 from the zero section. If the image of ψ_0 were to intersect the zero section, then all $\{\psi_m\}$ would as well. But, the latter is outlawed by (5.3.1). Second, ψ_0 is a covering map onto its image. This is because the projection from N_C to C is pseudo- holomorphic for the almost complex structure in (2.5) where $k \equiv 0$. Third, let $C_0 \subset N_C$ denote the image of ψ_0. As the projection from N_C to C induces a covering map from C_0 to C, it follows that the intersection of C_0 with any two fibers of N_C have the same number of points, which is the degree of the covering map from C_0 to C.

With the third point above understood, let z be any point in C. Then there is a neighborhood $V \subset C$ of z over which C_0 is given by the image of a set $(s_1, \ldots, s_p)$ of some $p \leq n$ sections of N_C, where each is annihilated by the operator D. Here, p is the degree of the projection induced map from C_0 to C. As z varies over C, these p sections define a non-trivial element, s, in the kernel of the extension of D to some $V_\rho \otimes N$ for some representation ρ as described in Definition 4.1.

Next, note that $p = 1$ is not allowed, for in this case, the torus C would be degenerate (as in Definition 2.1). As remarked above, when

(J, Ω) are chosen from an appropriate dense, open set, one can assume that C is not degenerate. For the same reason, one can assume that the $\mathbb{Z} \oplus \mathbb{Z}$ action on kernel(D) does not fix s. It follows from the definition of p that $\mathbb{Z} \oplus \mathbb{Z}$ acts transitively on the set $\{1, \ldots, p\}$, as C_0 is connected. The definition of p also requires that O_s have dimension 2 or more when $p > 2$.

Proof of Lemma 5.4. First of all, the case $n = 1$ follows from previous arguments. Now, if a square zero, embedded, pseudo-holomorphic torus C is non-degenerate in the sense of Definition 2.1, then whenever J_1 is sufficiently close to J, there will be a nearby J_1 pseudo-holomorphic torus for any sufficiently small perturbation of the given J. A corrected version of the argument in the final paragraph of Section 7b in [11] shows the following: Given C as above, there are arbitrarily small perturbations of J which produce nearby pseudo-holomorphic tori which are n-non-degenerate for any given n. As noted, the argument in the final paragraph of Section 7b in [11] is incorrect. To correct the argument, allow t to very in $\mathbb{R}$. Then M_t has real analytic dependence on t, and so its determinant either vanishes for all t, or only on a discrete set of t.

Given these last facts, the lemma now follows from the following observations:

(5.8)

1. $\mathcal{K}(J, \Omega)$ is, in all cases, locally compact.
2. If D has no kernel on a given V_ρ for given (ν, μ), then there is a C^0 open neighborhood of (ν, μ) where the corresponding D also has no kernel.

Note that Lemma 5.4 follows readily from Lemmas 5.12 and 5.13, below. The arguments given below for these lemmas are fundamentally different from that just given.

Addendum. There are only three issues in the preceding proof that do not come, more or less, from Section 5 of [7]. The first concerns the expicit focus here on embeddings of Σ into X. This focus is justified by the extra observation that there is a universal model as in (5.6), which has $p \neq 0$, for non-embedded pseudo-holomorphic maps, and that for these universal models, -2 is an upper bound to the index of the differential of the tautological map to $\mathcal{A}_d$. With this understood, the Sard-Smale theorem implies that the appearance of non-embedded pseudo-holomorphic maps in $\mathcal{K}(\cdot)$ as (J, Ω) vary is a codimension-two phenomena.

The second issue from Proposition 5.2 which is not covered in [7] is Lemma 5.4 which facilitates the arguments for the multiply toroidal cases.

The third issue concerns the fact that the set $\mathcal{U}$ in Proposition 5.2 is dense and open rather than just Baire. This conclusion can be drawn from the following facts: First, the moduli spaces of pseudo-holomorphic maps are locally compact. Second, one can invoke the Gromov compactness theorem to conclude that only finitely many such spaces need be considered when studying either of the following two cases:

1. The possible behavior of sequences in any given $\mathcal{K}(J, \Omega)$.

2. The behavior of sequences whose k'th element is chosen from the k'th element of a sequence of spaces $\{\mathcal{K}(J_m, \Omega_m)\}$ where the sequence $\{(J_m, \Omega_m)\}$ is constrained to converge to a given (J, Ω).

Third, as (ν, μ) and the complex structure on the surface Σ vary, the condition that the operator in (2.8) be surjective defines an open set in this parameter space.

c) Proof of Proposition 4.3, Part 2

This step completes the proof of Proposition 4.3. First of all, the assertion that the set of admissable (J, Ω) is Baire is an immediate consequence of Proposition 5.2. (The countable intersection of open and dense or Baire sets is itself Baire.)

Assertions 2 and 3 of Proposition 5.2 follow from the fact that the set admissable (Ω, J) is Baire. Thus, it only remains to prove Assertion 1. And, this assertion follows from Lemma 5.5, below, using the aforementioned fact that the set of admissable (Ω, J) is Baire.

Lemma 5.5. *Given $e \in H^2(X; \mathbb{Z})$, there is a Baire subset of $\mathcal{A}_d$ such that when (J, Ω) is chosen from this set, then there are but finitely many classes in $H^2(X; \mathbb{Z})$ which can be Poincaré dual to the fundamental class of a pseudo-holomorphic submanifold appearing (with some multiplicity) as an element in some $\{(C_k, m_k)\}$ from $\mathcal{H}(e, J, \Omega)$.*

Proof of Lemma 5.5. There are two steps to the proof. The first part derives an apriori upper bound on the genus of any connected component of a pseudo-holomorphic submanifold $C \subset \mathcal{H}(e, J, \Omega)$ under the assumption that (J, Ω) is suitably generic. The second step deduces the conclusions of Lemma 5.5 from the apriori genus bound.

Step 1. Fix $\{(C_k, m_k)\} \in \mathcal{H}(e, J, \Omega)$. Let e_k denote the Poincaré dual to the fundamental class of C_k. Since the C_k are pairwise disjoint, so the e_k are pairwise orthogonal with respect to the cup product pairing. This implies that

$$e \cdot e = \sum_k e_k \cdot e_k. \tag{5.9}$$

Also, by linearity,

$$c \cdot e = \sum_k c \cdot e_k, \tag{5.10}$$

where $c = c_1(K)$.

With the preceding understood, it follows that the integer d (as defined from e in (1.3)) is equal to the sum of the corresponding d_k (each defined by (1.3) but with e_k replacing e). According to Proposition 5.2, if (J, Ω) is chosen from an appropriate Baire subset, then all of these d_k are non-negative. Since they sum to d, one is forced to conclude that $0 \leq d_k \leq d$.

Write the genus of C_k using (1.2) and the e_k version of (1.3) as

$$g_k = 1 + c \cdot e_k - d_k. \tag{5.11}$$

This last equation bounds $c \cdot e_k$ from below by $-1 + g_k + d_k$. Note that this last number is non-negative unless $d_k = g_k = 0$. That is, unless e_k is Poincaré dual to a pseudo-holomorphic 2-sphere with self-intersection number -1. The maximum number, n, of such disjoint 2-spheres is apriori bounded by the homeomorphism type of X since each such 2-sphere corresponds to a decomposition of X as a connect sum with a $\mathbb{CP}^2$ with its non-complex orientation.

With n understood, then (5.10) bounds $c \cdot e_k$ by $c \cdot e + n$. Put this last bound back in (5.11) to bound g_k by $1 + c \cdot e + n$.

Step 2. One further equality is required in order to complete the proof of Lemma 5.5. To state this equality, let $[\omega]$ denote the cohomology class of the symplectic form ω. By linearity,

$$[\omega] \cdot e = \sum_k [\omega] \cdot e_k. \tag{5.12}$$

Since each e_k is Poincaré dual to a pseudo-holomorphic curve, the cup product of $[\omega]$ with e_k is positive. Then, (5.12) bounds this cup product:

$$[\omega] \cdot e \geq [\omega] \cdot e_k > 0. \tag{5.13}$$

Now, let $\{e_1, \dots\}$ be an infinite set of cohomology classes such that each class in this set is Poincaré dual to a connected pseudo-holomorphic submanifold which appears (with some multiplicity) as an element in some $\{(C_k, m_k)\}$ from $\mathcal{H}(e, J, \Omega)$. (Of course, there is no requirement that these e_k sum to e.) Lemma 5.5 follows from a demonstration that $\{e_k\}$ has only finitely many distinct elements. Indeed, suppose not. Then one can assume, with out loss of generality that no two elements are the same. This assumption generates a contradiction as follows: According to Step 1, there is an infinite subsequence, hence relabled sequentially from 1, with the property that for each k, the genus g_k, as defined in (1.2) with e replacing e_k, is equal to some fixed $g \geq 0$. Let $\{C_k\}$ denote the corresponding sequence of pseudo-holomorphic submanifolds. Think of each C_k as a complex curve with its tautological embedding $\varphi_k : C_k \longrightarrow X$. According to (5.13), there is a uniform bound on the energy of the sequence $\{\varphi_k\}$. With this undertood, one can appeal to the Gromov compactness theorem (as described, for example, in [6] and [15]) to describe the limiting behavior of the sequence $\{(C_k, \varphi_k)\}$. In particular, the Gromov compactness theorem implies that the sequence $\{e_k\}$ of cohomology classes has a convergent subsequence. Since these $\{e_k\}$ are integral valued classes, it follows that the convergent subsequence has only finitely many distinct elements, thus contradicting the initial assumption.

d) Proof of Theorem 1.1, Part 1.

The proof of Theorem 1.1 is reduced here to a proposition which concerns multiply toroidal classes and the numbers $\{r(C, m)\}$ of Definition 3.2. There are three steps in this reduction process.

Step 1. This first step introduces the invariant that Ruan defines in Section 5 of [7]. To begin, let $e \in H^2(X; \mathbb{Z})$ be a class which is not multiply toroidal. Suppose that a symplectic form ω has been specified. Then, choose (J, Ω) from the Baire subset $\mathcal{U}$ given by Proposition 5.2. It follows that the set $\mathcal{K}(J, \Omega)$ is finite and consists of non-degenerate (in the sense of Definition 2.1) pseudo-holomorphic submanifolds. As in [7], introduce

$$\mathrm{Ru}(e) \equiv \sum_C r(C, 1), \tag{5.14}$$

where the sum is over the points $C \in \mathcal{K}(J, \Omega)$, and $r(C, 1) = \pm 1$ is defined in (2.13). As asserted in Section 5 of [7], this number depends only on the class e and the deformation class of the symplectic form ω.

The invariance of (5.14) follows directly from Assertion 3 of Proposition 5.2.

Step 2. The purpose of this step is to define an analog of (5.14) for multiply toroidal classes. For this purpose, suppose that $e \in H^2(X; \mathbb{Z})$ is an indivisable class which obeys $e \cdot e = 0$ and $c \cdot e = 0$. Call such a class indivisably toroidal. Let $n \geq 1$ be an integer. According to Proposition 5.2, there is a Baire subset of ω-compatible, almost complex structures J which have the following properties:

(5.15)

1. When $m \leq n$ is a positive integer, then the set of connected, pseudo-holomorphic submanifolds with fundamental class Poincaré dual to $m \cdot e$ contains only finitely many elements.

2. Each element of one of these sets is n-non-degenerate in the sense of Definition 4.1.

3. The number of elements in each of these sets remains unchanged when J is perturbed slightly in the space of ω-compatible, almost complex structures.

With the preceding understood, choose the almost complex structure J from the afore-mentioned Baire subset. Now, define

$$\text{Qu}(e, n) \equiv \sum_{\{(C_k, m_k)\}} \prod_k r(C_k, m_k), \tag{5.16}$$

where the sum is over all sets $\{(C_k, m_k)\}$ which obey the following:

(5.17)

1. C_k is an embedded, pseudo-holomorphic torus in X whose fundamental class is Poincaré dual to $q_k e$ for some positive integer $q_k \leq n$.

2. Each m_k is a positive integer and $\sum_k q_k m_k = n$.

Here, $r(C, m)$ is given in Definition 3.2. Note that the sum in (5.16) is finite thanks to (5.15). (By the way, as e is not multiply toroidal, then $\text{Ru}(e)$ is well defined; it equals $\text{Qu}(e, 1)$.)

For purposes to come, it proves convenient to introduce a generating functional to keep track of the data $\{r(C,m)\}_{m>0}$ when C is an embedded, pseudo-holomorphic torus in X whose fundamental class is Poincaré dual to $q \cdot e$ for some $q > 0$. This generating functional $P(C;z)$ is a formal power series in the indeterminant z whose constant term is 1 and m'th coefficient is $r(C,m)$. That is,

$$P(C;z) \equiv 1 + \sum_{m \geq 1} r(C,m) \cdot z_m. \tag{5.18}$$

Note that $\mathrm{Qu}(e,n)$ from (5.16) is the coefficient of z_n in the formal power series $\prod_k P(C_k, z)$.

Step 3. Let $e \in H^2(X;\mathbb{Z})$. This step rewrites $\mathrm{Gr}(e)$ in terms of the invariant $\mathrm{Ru}(\cdot)$ in (5.14) and the numbers $\mathrm{Qu}(\cdot)$ that were just defined. With the preceding understood, let $S(e)$ denote the collection of sets of unordered pairs of the form $\{(e_k, n_k)\}$ where the following constraints are imposed:

(5.19)

1. $\{e_k\} \subset H^2(X;\mathbb{Z})$ is a set of distinct, non-multiply toroidal classes.
2. $n_k = 1$ unless $e_k \cdot e_k = 0$ in which case n_k can be any positive integer.
3. $e_k \cdot e_m = 0$ if $k \neq m$.
4. $e = \sum_k n_k e_k$.

Given some $y = \{(e_k, n_k)\} \in S(e)$, let $\tau(y)$ denote the set of those pairs (e_k, n_k) which appear in y and obey at least one of the following two conditions:

(5.20)

1. $e_k \cdot e_k \neq 0$.
2. $c \cdot e_k \neq 0$.

With the preceding understood, consider

Lemma 5.6. *Let $e \in H^2(X;\mathbb{Z})$. Then* $\mathrm{Gr}(e)$ *in (1.7) is equal to*

$$\mathrm{Gr}(e) = (d!) \sum_{y \in S(e)} \left(\prod_{(e_k,n_k)\in\tau(y)} \frac{1}{(d_k!)^{n_k}(n_k!)} [\mathrm{Ru}(e_k)]^{n_k} \right) \times \left(\prod_{(e_k,n_k)\notin\tau(y)} \mathrm{Qu}(e_k,n_k) \right). \tag{5.21}$$

Proof of Lemma 5.6. This is a simple resummation of (1.7).

Since $\mathrm{Ru}(e)$ is already known to be a deformation invariant of the symplectic form X, Theorem 1.1 follows immediately from Lemma 5.6 and

Proposition 5.7. *Let $e \in H^2(X;\mathbb{Z})$ be an indivisable class which obeys $e \cdot e = 0$ and also $c \cdot e = 0$. Let $n \geq 1$ be an integer. Then* $\mathrm{Qu}(e,n)$ *as defined above depends only on the deformation equivalence class of the symplectic form ω.*

e) Proof of Proposition 5.7.

The purpose of this subsection is to reduce the proof of Proposition 5.7 to that of four basic lemmas about the space $\mathcal{X}_\gamma$ of Proposition 5.2. Four steps deduce Proposition 5.7 from these basic lemmas. The proofs of the four basic lemmas occupy the next four subsections.

Step 1. To begin the discussion suppose that ω_0 and ω_1 are a pair of symplectic forms that are connected by a path $\{\omega_t : t \in [0,1]\}$ of such forms. For each positive integer $m \leq n$, use Proposition 5.2 to find a section $\gamma : [0,1] \longrightarrow \mathcal{A}_0$ so that the corresponding $\mathcal{X}_\gamma$ from (5.4) for the class $m \cdot e$ is an oriented, 1-dimensional manifold. (The Sard-Smale argument which finds γ allows one to choose a fixed section γ which works simultaneously for all choices of the integer m.) So as not to confuse the different versions of (5.4) for different values of m, introduce the notation $\mathcal{X}_{\gamma,m}$ to denote the $m \cdot e$ version of (5.4).

With m fixed, let $\mathcal{X} \equiv \cup_{m_1 \leq m} \mathcal{X}_{\gamma,m_1}$, topologized as the disjoint union. None-the-less, say that $(t,C) \in \mathcal{X}$ is a weak limit of a sequence $\{(t_k,C_k)\} \subset \mathcal{X}$ if the sequence $\{t_k\}$ converges to t, and the sequence $\{C_k\}$ with their embeddings $\{\psi_k : C_k \longrightarrow X\}$ converges in the following sense: First, $\{C_k\}$ converges to a complex torus C_0. Second, $\{\psi_k\}$ converges to a pseudo-holomorphic map $\psi_0 : C_0 \longrightarrow X$ which factors as $\psi \circ f$, where $f : C_0 \longrightarrow C$ is a holomorphic covering map, and

$\psi : C \longrightarrow X$ is a pseudo-holomorphic embedding with the almost complex structure on X defined by the image of $\gamma(t)$.

For a poorly chosen class γ, this $\mathcal{X}$ can be quite complicated. The following sequence of lemmas detail the relevant structure of $\mathcal{X}$ for a reasonable γ. The proofs of these lemmas are provided in later subsections.

The statement of the first lemma below introduces the canonical map (induced by projection) $\pi : \mathcal{X} \longrightarrow [0, 1]$ which is defined as follows: A point in $\mathcal{X}$ has the evident form (t, C), where $t \in [0, 1]$, and $C \subset X$ is an embedded torus. Then, $\pi(t, C) = t$.

Lemma 5.8. *The path γ can be chosen so that $\mathcal{X}$ has the following properties:*

1. *$\mathcal{X}$ is a 1-dimensional manifold which consists solely of points (t, C) where $C \subset X$ is an embedded torus.*

2. *The critical points of π are each non-degenerate.*

3. *The set of weak limit points of sequences in $\mathcal{X}$ is disjoint from both 0, 1 and from the set of critical points of π.*

4. *The set of weak limit points and the set of critical points are both finite sets.*

5. *If $t \in (0, 1)$, and $\pi^{-1}(t)$ contains neither critical points, nor weak limit points, then $\pi^{-1}(t)$ is a finite set whose elements are n-non-degenerate. Furthermore, the almost complex structure indicated by $\gamma(t)$ is such that (5.15) is obeyed and so the pseudo-holomorphic tori in $\pi^{-1}(t)$ can be used in (5.16) for the computation of* $\mathrm{Qu}(e, n)$.

6. *If $\{(t_m, C_m)\}$ is a sequence in $\mathcal{X}$ with no convergent subsequences, then there is a subsequence which has a weak limit point in $\mathcal{X}$.*

The next lemma describes the structure of $\mathcal{X}$ near the critical points of π. This lemma explicitly re-introduces the operator D from Section 2 and its twisted cousins $\{D_\iota : \iota \in H^1(C; \mathbb{Z}/2)\}$ from Section 3. (The sign of the determinants of these operators are defined as in (2.13).)

By the way, here is a baby model for the situation described below: Let $h(\epsilon, \alpha)$ be a smooth proper function on $(-1, 1) \times \mathbb{R}$. Model γ by $h^{-1}(0)$, and model π as the map to $(-1, 1)$. The partial derivative of h

with respect to α corresponds to the operator D, and the sign of this derivative corresponds to the weight $r(C,1)$. In this case, the generic behavior near a critical point of π is given by the special case where $h(s,t) = \alpha^2 - \epsilon$. In this model, γ is empty where $\epsilon > 0$, and γ consists of two components where $\epsilon < 0$; and these are weighted with opposite signs.

Lemma 5.9, below, asserts that this special model situation also describes the local structure of $\mathcal{X}$ near a critical point of π.

Lemma 5.9. *The conclusions of Lemma 5.8 can be amended to include the following: Let $(t,C) \in \mathcal{X}$ be a critical point of the map π. Then there exists an open interval $\mathcal{J}$ of the form $(t-\epsilon,t)$ or else $(t+\epsilon,t)$; and there exists a pair of maps $\lambda_\pm : \mathcal{J} \longrightarrow \mathcal{X}$ with the following properties:*

1. $\pi \circ \lambda_\pm =$ *identity.*
2. $\mathrm{Image}(\lambda_+)$ *is disjoint from* $\mathrm{Image}(\lambda_-)$.
3. $\lim_{s\to t} \lambda_\pm(s) = (t,C)$.
4. $\mathrm{Image}(\lambda_+) \cup \mathrm{Image}(\lambda_-) \cup (t,C)$ *is an open neighborhood of* (t,C).
5. *Write $\lambda_\pm(s) = (s, C_\pm)$. Then the sign of* $\det(D)$ *for C_+ is positive and that for C_- is negative. However, for each $\iota \neq 0$, the corresponding signs of* $\det(D_\iota)$ *agree for C_+ and C_-.*

The following lemma considers the structure of $\mathcal{X}$ near the ends of $\mathcal{X}$ as defined by sequences without convergent sequences. Here is the baby model in this case: Let $h(\epsilon,\alpha)$ be a smooth, proper function on $(-1,1)\times\mathbb{R}$ which obeys $h(\epsilon,-\alpha) = -h(\epsilon,\alpha)$. The model for γ is the union of $(-1,1)\times\{0\}$ with $\{(\epsilon,\alpha) : h(\epsilon,\alpha) = 0 \text{ and } \alpha \neq 0\}/\sim$, where $(\epsilon,\alpha)\sim(\epsilon,-\alpha)$. The model behavior occurs for the case where $h = \alpha\cdot(\epsilon-\alpha^2)$. The component $(-1,1)\times\{0\}$ corresponds to a torus in the class e, and the other component corresponds to a torus in the class $2\cdot e$. The point $(0,0)$ corresponds to a weak limit point. Note that when $\epsilon > 0$, there is no $2\cdot e$ component, while for $\epsilon < 0$, there is one such component. This example illustrates how an e-component can affect the count of a $2\cdot e$ component. The α derivative of h models D_ι on the $(-1,1)\times\{0\}$ component of γ, and models D on the other component of γ.

Lemma 5.10. *The conclusions of Lemma 5.8 can be amended to include the following: Let (t,C) be a weak limit point in $\mathcal{X}$.*

1. *There is an interval* $\mathcal{J}$ *which is either* $(t-\epsilon, t)$ *or else* $(t+\epsilon, t)$ *for some* $\epsilon > 0$*; and there is a map* $\tau : \mathcal{J} \longrightarrow \mathcal{X}$ *which obeys:*
   a) $\pi \circ \tau =$ *identity,*
   b) (t, C) *is not a weak limit point of any sequence in* $\mathcal{X} - \tau(\mathcal{J})$.
2. *If* $(t, C) \in \mathcal{X}_{\gamma,m}$*, then* $\tau(\mathcal{J}) \subset \mathcal{X}_{\gamma,2m}$.
3. *If* $\{(t_k, C_k)\} \subset \tau(\mathcal{J})$ *is a non-convergent sequence, then* $\{t_k\}$ *converges to* t *and* $\{C_k\}$ *converges to a complex torus* C_0 *while the sequence of embeddings* $\{\psi_k : C_k \longrightarrow X\}$ *converges to a map* ψ_0 *of the form* $\psi \circ f$*, where* f *is a non-trivial, 2 to 1 covering map of* C_0 *onto a complex torus,* C*, and* ψ *is the tautological embedding of* C *into* X.
4. *There is a map* $\lambda : [t-\epsilon, t+\epsilon] \longrightarrow \mathcal{X}_{\gamma,m}$ *with* $\pi \circ \lambda =$ *identity and such that* $\lambda(t) = (t, C)$.
5. *Write* $\lambda(s) = (s, C')$*. Let* $\iota \in H^1(C; \mathbb{Z}/2)$*. Then the operator* D_ι *for* C' *has trivial kernel unless* ι *classifies the covering map* f*. In this case, the kernel is trivial except when* $s = t$ *(so* $C' = C$*).*
6. *Furthermore, when* ι *classifies the map* f*, then the sign of* $\det(D_\iota)$ *changes as* s *passes* t*. For the other choices of* ι*, the sign of* $\det(D_\iota)$ *is independent of* s.

Note that each torus in the image of the map τ from Lemma 5.8 has its operator D and thus an associated ± 1 from sign(det(D)) as defined in (2.13). This sign can be determined in terms of the sign of the determinants of the operators D and D_ι for a torus in the $\lambda-$image of $\mathcal{J}$ (but in particular, not C). Here, the subscript ι denotes the element in $H^1(C; \mathbb{Z}/2)$ that classifies the covering map f. One can also determine the signs of the twists of D for tori in the image of τ. In this regard, remark that the 2 to 1 covering map f induces

$$f^* : H^1(C; \mathbb{Z}/2) \longrightarrow H^1(C'; \mathbb{Z}/2), \tag{5.22}$$

which is 2 to 1 onto its image. Note that f^* is linear and sends ι to zero when ι classifies f.

With the preceding understood, consider:

Lemma 5.11. *The conclusions of Lemma 5.10 can be amended to include the following:*

1. *Let* $\iota \in H^1(C;\mathbb{Z}/2)$ *classify the covering map* f. *Then the sign of* $\det(D)$ *for a torus in* $\tau(\mathcal{J})$ *is determined from the signs of* $\det(D_\iota)$ *and* $\det(D)$ *from a torus in* $\lambda(\mathcal{J})$ *by the rule*

$$\begin{aligned}&\operatorname{sign}(\det(D))|_{\mathrm{Image}(\tau)}\\&\quad= -\operatorname{sign}(\det(D_\iota))|_{\mathrm{image}(\lambda)}\cdot\operatorname{sign}(\det(D))|_{\mathrm{image}(\lambda)}.\end{aligned}$$

2. *Let* $\iota' \in H^1(C';\mathbb{Z}/2)$ *denote the non-zero class in the image of* f^*. *Then*

$$\operatorname{sign}(\det(D'_\iota))|_{\mathrm{Image}(\tau)} = \prod_{j:f^*(j)=\iota'} \operatorname{sign}(\det(D_j))|_{\mathrm{image}(\lambda)}.$$

3. *Let* $\iota' \in H^1(C';\mathbb{Z}/2)$ *be any class not in the image of* f^*. *Then*

$$\operatorname{sign}(\det(D'_\iota))|_{\mathrm{Image}(\tau)} = 1.$$

Step 2. According to Assertion 3 of Lemma 5.8, the pseudo-holomorphic tori in $\pi^{-1}(t)$ can be used to compute $Q(e,n)$ when t is neither a critical value of π, nor a limit point of the π-image of a non-convergent sequence. In the analysis which follows, it will be assumed that the critical points of π have distinct critical values, and that these are discrete from the finite set of times t which label (in part) a weak limit point $(t, C) \in \mathcal{X}$. In the subsequent steps, the latter set of times will also be assumed distinct. These assumptions are for convenience only, and the arguments below are only notationally more complicated when these assumptions are relaxed. The arguments without these assumptions are left to the reader. Note that one can also prove that the aforementioned times t can be made distinct by perturbing γ.

Suppose that t is a critical value of π. Then, for all $\epsilon > 0$ but small, one can use either $\pi^{-1}(t \pm \epsilon)$ to compute $\mathrm{Qu}(e,n)$. The purpose of this step is to establish that the computation of $\mathrm{Qu}(e,n)$ using $\pi^{-1}(t-\epsilon)$ gives the same answer as that using $\pi^{-1}(t+\epsilon)$.

To begin, let (t, C) be the relevant critical point. If this point (t, C) lies in $\mathcal{X}_{\gamma,n}$, then Assertion 5 of Lemma 5.9 impies that the before t computation and after t computations of $\mathrm{Qu}(e,n)$ give the same answer. (Either one +1 and one -1 disappear from the sum in (5.16) as ϵ crosses zero, or else one +1 and one -1 appear.)

To argue in the general case, assume first that $\mathcal{J} = (t-\epsilon, t)$. The argument for the other case is essentially identical and left to the reader.

Suppose also that $(t, C) \in \mathcal{X}_{\gamma,m}$ for some $m \leq n$ which divides n. Let $\mathrm{Qu}_-(e,n)$ denote the value of (5.16) as computed with $\pi^{-1}(t-\epsilon)$ and let $\mathrm{Qu}_+(e,n)$ denote the analogous sum as computed with $\pi^{-1}(t+\epsilon)$. As in Lemma 5.9, write $\lambda_\pm(t-\epsilon_1) = (t-\epsilon_1, C_\pm)$ when $\epsilon_1 \in (0,\epsilon)$. Then

$$
\begin{aligned}
&\mathrm{Qu}_+(e,n) - \mathrm{Qu}_-(e,n) \\
&\quad = - \sum_{(m_1,m_2)} r(C_+, m_1)\cdot r(C_-, m_2) \\
&\qquad\qquad \cdot \mathrm{Qu}_+(e, n - m\cdot m_1 - m\cdot m_2),
\end{aligned}
\tag{5.23}
$$

where the sum is over all pairs of non-negative integers (m_1, m_2) with the property that $0 < (m_1 + m_2)\cdot m \leq n$. To put this last formula into perspective, introduce the formal power series $P(C, z)$ from (5.18), and also

$$
Q_\pm[z] \equiv 1 + \sum_{p\geq 0} \mathrm{Qu}_\pm(e,p)\cdot z^p. \tag{5.24}
$$

Now, reintroduce the map δ in (3.1). As $P(C; z)$ depends only on the map δ, it proves useful to exploit this fact by introducing the formal power series $\{P_{\pm k} : k = 0,1,2,3\}$ where $P_{+k} = P(C;\cdot)$ in the case where the map δ for C sends the trivial element to 1, and precisely k other elements to -1. Likewise, $P_{-k} = P(C;\cdot)$ in the case where the map δ for C sends the trivial element and precisely k other elements to -1.

With the preceding understood, (5.22) implies that

$$
Q_+(z) - Q_-(z) = (1 - P_{+k}(z^m)\cdot P_{-k}(z^m))\cdot Q_+(z). \tag{5.25}
$$

(This makes use of Assertion 5 of Lemma 5.9.) If $Q_+(z)$ and $Q_-(z)$ are equal, it must be the case that

$$
P_{+k}(z) = 1/P_{-k}(z). \tag{5.26}
$$

The reader can verify (5.26) using (3.2) and Assertion 5 of Lemma 5.9.

Step 3. Now suppose that t is the limit point of the π-image of a non-convergent sequence in $\mathcal{X}$. As in Lemma 5.10, introduce the interval $\mathcal{J}$ and the map τ. Suppose first that $\mathcal{J} = (t+\epsilon, t)$. Write $\tau(s) = (s, C_0)$ and write $(s) = (s, C_+)$ when $s > t$. And, write $(s) = (s, C_-)$ when $s < t$. Introduce, as before, $Q_\pm(t)$. Also, introduce the formal power series $P(C_0; z)$ and $P(C_\pm, z)$. Then, (5.16) implies that Q_+ and Q_- are related by

$$Q_+ - Q_- = (P(C_+; z^m) \cdot P(C_0; z^{2m})/P(C_-; z^m) - 1) \cdot Q_-. \tag{5.27}$$

Thus, if $Q_\pm$ are equal, the forced conclusion is that

$$P(C_+; z) = P(C_-; z)/P(C_0; z^2). \tag{5.28}$$

Step 4. This step uses Lemma 5.11 and (5.28) to further constrain the possibilities for $P_{\pm k}$. Indeed, Lemma 5.11 with (5.26) and (5.28) determine all 16 possibilities for $P_{\pm k}$ in terms of any one, in particular $P_{\pm 0}$. To prove this assertion, consider (5.28) where δ for C_- has image equal to $+1$. Then, δ for C_+ will map the trivial element to $+1$ and exactly one non-trivial class ι to -1. (This is the class which classifies the 2 to 1 covering.) According to Lemma 5.11, the image of δ for C_0 is solely the trivial element. This implies that

$$P_{+1}(z) = \frac{P_{+0}(z)}{P_{+0}(z^2)}. \tag{5.29}$$

For the next case, suppose that δ for C_- sends the trivial element to 1 and exactly one element to -1, while δ for C_+ sends the trivial element to 1 and exactly two elements to -1. According to Lemma 5.11, δ for C_0 will send the trivial element to 1 and exactly one element to -1. Thus, (5.28) reads

$$P_{+2}(z) = \frac{P_{+1}(z)}{P_{+1}(z^2)} = \frac{P_{+0}(z)P_{+0}(z^4)}{(P_{+0}(z^2))^2}. \tag{5.30}$$

Finally, suppose that δ for C_- sends the trivial element to 1 and exactly two elements to -1, while δ for C_+ sends the trivial element to 1 and all three other elements to -1. According to Lemma 5.11, δ for C_0 will send all elements to 1. Thus (5.28) reads

$$P_{+3}(z) = \frac{P_{+2}(z)}{P_{+0}(z^2)} = \frac{P_{+0}(z)P_{+0}(z^4)}{(P_{+0}(z^2))^3}. \tag{5.31}$$

Given (5.29-31), one can use (5.26) to determine $P_{-k}(z)$. It is left to the reader to check that no other constraints on $P_{\pm k}$ are introduced by the other possibilities for δ on C_+ and on C_-. The reader can also verify (5.29-31) and (5.26) for (3.2) with the choice

$$P_{+0} = \frac{1}{1-z}. \tag{5.32}$$

No new information about e comes from a different choice for P_{+0} as long as the new choice begins as $1 + z + \mathcal{O}(z^2)$.

f) Proof of Assertions 1, 2 of Lemma 5.8 plus Lemma 5.9. The proofs begin with a digression to discuss some specific genericity assumptions that can be imposed on the space $\mathcal{X}$. (These genericity assumptions are proved by arguments which involve only minor modifications to those which proved Assertion 3 of Proposition 5.2.)

To begin the digression, recall that the complex structures on the torus are parameterized by a point in the quotient of $\mathbb{C}_+ = \{\tau \in \mathbb{C} : \operatorname{im}(\tau) > 0\}$ by the action of the group $\mathrm{Sl}(2;\mathbb{Z})/\{\pm 1\}$. This action is free on the compliment of a countable set of points. Let $\mathbb{C}_{++} \subset \mathbb{C}$ denote the subset where this action is free. With the preceding understood, remark that there is a natural map $q : \mathcal{X} \longrightarrow \mathbb{C}_+ / \mathrm{Sl}(2;\mathbb{Z})$ and one can choose the path γ so that the image of q lands in $\mathbb{C}_{++} / \mathrm{Sl}(2;\mathbb{Z})$. This last point is proved using the fact that $\mathbb{C}_+ \backslash \mathbb{C}_{++}$ has codimension 2 in the arguments for Assertion 3 of Proposition 5.2.

Now introduce the vector bundle $\mathcal{G} \longrightarrow \mathbb{C}_{++} / \mathrm{Sl}(2;\mathbb{Z})$ whose fiber at a given orbit is the vector space of pairs (ν, μ) of complex valued functions on the complex curve which the orbit parameterizes. Associated to each $(t, C) \in \mathcal{X}$ is the operator D as in (2.7); and with this point understood, the coefficients of D define a natural map $q_0 : \mathcal{X} \longrightarrow \mathcal{G}$ which covers the map q.

It proves useful in subsequent arguments to be able to assume that q_0 is transversal to certain subvarieties in $\mathcal{G}$. Here, a subvariety means a subspace $\mathcal{D}$ which is a countable, disjoint union $\cup_k \mathcal{D}^k$ of submanifolds which are constrained by the following two requirements. First, $\mathcal{D}^k$ always has codimension k. Second, $\cup_{k \geq k_0} \mathcal{D}^k$ is always closed.

The following lemma is proved by making some minor modifications to the fiber product set up that is used in the proof of Assertion 3 of Proposition 5.2. As with Assertion 3, the result follows in the end from the Sard-Smale theorem.

Lemma 5.12. *Let $\mathcal{D} \subset \mathcal{G}$ be a fixed subvariety. Then γ can be chosen in Proposition 5.2 so that the map q_0 is transversal to the component submanifolds of $\mathcal{D}$. In particular, q_0 misses $\cup_{k \geq 2} \mathcal{D}^k$ and is transversal to $\mathcal{D}^1$.*

With this lemma understood (the proof is left to the reader), note that the proofs of Lemmas 5.8 and 5.9 invoke this lemma for a very specific set of varieties. The relevant varieties are described in

Lemma 5.13. *Introduce $\mathcal{G}$ as above.*

1. *Let $\mathcal{D} \subset \mathcal{G}$ denote the set of triples $(\tau, (\nu, \mu))$ for which the operator D_ι has a kernel for some $\iota \in H^1(C; \mathbb{Z}/2)$, where, the curve C is parametrized by the orbit of τ. Then $\mathcal{D}$ is a subvariety whose codimension-1 stratum consists of those triples where all but one D_ι has trivial kernel, and one D_ι has kernel dimension 1.*

2. *Fix a positive integer $n > 1$ and let $\mathcal{D}' \subset \mathcal{G}$ denote the set of triples $(\tau, (\nu, \mu))$ with the following property: There is a representation $\rho : \pi_1(C) \longrightarrow P_n$ and there is a section s of $V_\rho \otimes \mathbb{C}$ which is annihilated by D for which subspace $O_s \subset \mathrm{Kernel}(D)$ has dimension 2 or more. This $\mathcal{D}'$ is a subvariety with no codimension 0 or codimension 1 strata.*

The proof of this lemma is deferred to Subsection 5h. Given the preceding two lemmas, one can now require that

(5.33)
1. The map q sends $\mathcal{X}$ into $\mathbb{C}_{++} / \mathrm{Sl}(2; \mathbb{Z})$.
2. The map q_0 is transversal to all strata of the varieties $\mathcal{D}$ and $\mathcal{D}'$ of Lemma 5.13.

End the digression.

Proof of Assertions 1 and 2 of Lemma 5.8.

Assertion 1 of the lemma is directly a consequence of Proposition 5.2.

To prove Assertion 2, remark first that the proof of Assertion 3 of Proposition 5.2 shows that (t, C) is a critical point of the map π if and only if the operator D at C has non-trivial kernel. In this case, it follows from (5.33), that this kernel has dimension 1. Indeed, whether or not (t, C) is a critical point of π, perturbation theory can be used to describe a neighborhood $\mathcal{W} \subset \mathcal{X}$ of (t, C). This is done as follows: If $(t + \epsilon, C_\epsilon) \in \mathcal{X}$ is near to (t, C), the C_ϵ will be near to C and so can be expressed as the image of C by a section v_ϵ of the disk bundle $U \longrightarrow C$. The condition that C_ϵ be pseudo-holomorphic for the complex structure at $\gamma(t + \epsilon)$ at time $t + \epsilon$ translates into a non-linear differential equation for v_ϵ. This equation has the schematic form

$$Dv_\epsilon + \epsilon \cdot \Delta + R(\epsilon, v_s, \nabla v_s) = 0. \tag{5.34}$$

Here, $R(\epsilon, \cdot, \cdot)$ is, for $|\epsilon|$ small, a smooth, fiber preserving function from $N \oplus (T^*C \otimes N)$ to $T^{1,0}C \otimes N$ with $|R(\epsilon, v_1, v_2)|$ bounded by some uniform multiple of $(\epsilon^2 + |v_1|^2 + |v_2|^2)$. (This R is linear in its last argument.) Meanwhile, ∇ is the induced covariant derivative on sections of N, and Δ is a particular section of $T^{0,1} \otimes N$ which is proportional to the projection onto N of the restriction to $T_{0,1}C$ of the derivative at time t of the path, γ, of almost complex structures.

When kernel$(D) = \{0\}$, it is a straightforward exercise with the implicit function theorem to see that (5.34) has a unique small solution v_ϵ when ever ϵ is close to 0. (Here, small means small in the C^0 norm. Even so, all C^k norms of v_ϵ can be estimated uniformly in terms of k and $|\epsilon|$ when the C^0 norm is small.) See, e.g. the proof of Lemmas 5.4 and 5.5 in [11] for the details on setting up the implicit function theorem in a space of Hölder continuous functions. Note that v_ϵ will depend smoothly on ϵ for ϵ near 0. From this, one sees that (t, C) is not a critical point of π when D has trivial kernel.

Now suppose that kernel(D) is 1-dimensional, and spanned by s. In this case, cokernel(D) is 1-dimensional, and this gives an obstruction to solving (5.34). To analyze the obstruction, one writes $v_\epsilon = \alpha \cdot s + w$, where $\alpha \in \mathbb{R}$ is assumed small, and where w is L^2-orthogonal to s. Then, solve the projection of (5.34) orthogonal to cokernel(D) for w as a function of the pair (ϵ, α). The implicit function theorem finds a unique, small solution $w = w(\epsilon, \alpha)$ to this modified equation when (ϵ, α) are both small. Note that w is a smooth function of the pair (ϵ, α) whose size is $\mathcal{O}(\epsilon + |\alpha|^2)$.

With w understood to be a function of ϵ and α, project (5.34) onto the cokernel of D. Choose a non-zero element $s^* \in$ cokernel(D) and this projection defines a smooth function, g, on a neighborhood of the origin in $\mathbb{R}^2$ whose zero set is diffeomorphic to a neighborhood of (t, C) in $\mathcal{X}$. Here, note that the Taylor's expansion for g begins with

$$g = r_1 \cdot \epsilon + r_2 \cdot \alpha^2 + \dots , \tag{5.35}$$

where r_1 and r_2 are numbers. For example, r_1 is obtained by integrating $s^*\Delta$ over C.

With (5.35) understood, remark that r_1 is non-zero since $\mathcal{X}$ is known to be a manifold. And, r_2 is non-zero because q_0 (see (5.33) is assumed to be transversal to $\mathcal{D}$ at (t, C). In this case, the differential of the map F of Step 4 of the proof of Lemma 3.1 pulls back to $\mathcal{X}$ near (t, C) to be proportional to r_2.

Assertion 2 of Lemma 5.8 now follows from (5.35) with the observation above that neither r_1 nor r_2 is zero.

By the way, note that the preceding has identified the critical points of π with the q_0-inverse image of the subset of $\mathcal{G}$ for which the corresponding operator D has 1 dimensional kernel.

Proof of Lemma 5.9.

Assertions 1-4 of the lemma follow from the fact that a neighborhood of (t, C) in $\mathcal{X}$ has been proved to be diffeomorphic to a neighborhood of $(0,0)$ in the zero set of the function g given by (5.35). (Remember that neither r_1 nor r_2 is zero. Thus, α is a smooth parameter on $\mathcal{X}$ near (t, C), while ϵ is a smooth parameter on the complement of (t, C) in a neighborhood of (t, C) in $\mathcal{X}$.) As for Assertion 5, remark first that α in (5.35) defines a smooth parameter on a neighborhood in $\mathcal{X}$ of (t, C). With this understood, the sign difference for $\det(D)$ between C_+ and C_- follows from the fact that the function F in Step 4 of the proof of Lemma 3.1 restricts to the afore mentioned neighborhood of (t, C) so as to vanish at $\alpha = 0$ but have non-zero α derivative at $\alpha = 0$.

The lack of sign change for the determinant of D_ι when $\iota \neq 0$ can be argued as follows: The fact that q_0 is transversal to the variety $\mathcal{D}$ implies that $\text{kernel}(D_\iota)$ is trivial at C when $\iota \neq 0$. Since the absence of a kernel is a stable condition, this kernel is trivial for D_ι as defined for tori $C_\pm$ when $(s, C_\pm) \in \mathcal{X}$ is close to (t, C).

f) Proof of Assertions 3-6 of Lemma 5.8 and Lemma 5.10.

The proofs here require a four part digression to describe in greater detail the "ends" of $\mathcal{X}$ as defined by the behavior of sequences without convergent subsequences.

Part 1. To begin the digression, fix $(t, C) \in \mathcal{X}$ and a holomorphic torus C' together with a holomorphic covering map $f : C' \longrightarrow C$ of degree at least 2. Write ψ' for the composition of f with the tautological embedding $\psi : C \longrightarrow X$. Suppose now that there exist small ϵ and a map $\chi : C' \longrightarrow X$ which is close to $\psi \circ f$ and is a pseudo-holomorphic embedding as defined by the almost complex structure given by $\gamma(t+\epsilon)$. If χ is close to $\psi \circ f$, then its image will lie in the tubular neighborhood U, and χ is defined by a section $\underline{s}$ of f^*N. The condition that the image of χ be pseudo-holomorphic with respect $\gamma(t+\epsilon)$ translates into a differential equation for $\underline{s}$ of the form

$$D'\underline{s} + \epsilon \cdot f^*\Delta + (f^*R)(\epsilon, \underline{s}, \nabla'\underline{s}) = 0. \tag{5.36}$$

Here, D' is defined on the space of sections of f^*N over C' by (2.7) with the understanding that the ν and μ which appear are the f-pull-backs of those which define the operator D on C. Likewise, the covariant ∇' is defined in a natural way by the pull-back of data on C.

With (5.36) understood, remark that the group of deck transformations of f lifts in a natural way to an action on the bundle f^*N. It also lifts to a holomorphic action on $T^{1,0}C'$. Thus, the group of deck transformations of f acts linearly on the space of sections of f^*N and also on the space of sections of $T^{1,0}C' \otimes f^*N$. Furthermore, the operator D' which appears in (5.36) is equivariant with respect to this action, as is the covariant derivative ∇'. Also, $f^*\Delta$ in (5.36) is invariant under the group of deck transformation, and $R(\epsilon, \cdot, \cdot)$ is also well behaved.

The naturality of (5.36) with respect to the deck transformations can be exploited by decomposing the section s as a sum, $\underline{s} = s_0 + s_1$, where s_0 transforms trivially under the group of deck transformations (so is pulled up from C by f), and where s_1 averages to zero under the group of deck transformations. Then, split (5.36) into its deck invariant and non-invariant parts to obtain two equations:

$$(5.37)\qquad \begin{aligned} &1.\quad D's_0 + \epsilon \cdot f^*\Delta + \textstyle\prod_0 \cdot f^*R(\epsilon, s_0 + s_1, \nabla'(s_0 + s_1)) = 0. \\ &2.\quad D's_1 + \textstyle\prod_1 f^*R(\epsilon, s_0 + s_1) = 0. \end{aligned}$$

Here, $\prod_0$ is the L^2 orthogonal projection on C' onto the deck-invariant subspace of $C^\infty(C'; T^{1,0}C' \otimes f^*N)$, and $\prod_1$ is the complimentary projection. In particular, note that

$$(5.38)\qquad |\textstyle\prod_1 f^*R(\epsilon, s_0 + s_1)| \le z \cdot (|s_1| + |\nabla s_1|) \cdot (\epsilon + |s_0| + |s_1|),$$

where z is a C-dependent constant. (This follows from the fact that $\prod_1 f^*R$ must vanish when $s_1 \equiv 0$.)

The focus on (5.36-38) here is justified by

Lemma 5.14. *Let $\{(t_k, C_k)\} \subset \mathcal{X}$ be a sequence with no convergent subsequence. Then there is an infinite subsequence with a weak limit point, $(t, C) \in \mathcal{X}$. Furthermore, there exists a complex torus C' with a holomorphic covering map $f : C' \longrightarrow C$ of degree at least 2 with the following significance: Given $\delta > 0$, there exists, for all but finitely many indices k, a section $\underline{s}$ of f^*N, which is unique up to the action of the deck transformations of f, and which has the three properties listed below:*

1. $|\underline{s}| < \delta$,

2. *$\underline{s}$ obeys (5.36) with $\epsilon = t_k - t$,*

3. *The push-forward of the image of $\underline{s}$ to the tubular neighborhood U of C embeds the torus C' in X as the torus C_k.*

Note that the embedding in Assertion 3 is not generally pseudo-holomorphic with respect to the given complex structure on C'.

Proof of Lemma 5.14. Repeat the arguments which prove Lemma 5.3.

Part 2. This part begins the analysis of (5.37). Consider first

Lemma 5.15. *Suppose that the kernel of D' consists solely of deck invariant elements. Then, when $|\epsilon|$ is small, any solution $\underline{s}$ to (5.37) with small C^0-norm has $s_1 \equiv 0$ and is thus the f-pull back to C' of a solution to (5.34) on C.*

Proof of Lemma 5.15. Under the assumptions of the lemma, the L^2 norm of $D's_1$ on C' is greater than some non-zero multiple λ of the sum of the L^2_1 norm of s_1. However, according to (5.37.2) and (5.38), the L^2 norm of $D's_1$ is smaller than a multiple, say z', of the sum of L^2_1 norm of s_1. Here, z' is itself bounded by a multiple of the larger of $|\epsilon|$ and the C^0 norm of $\underline{s}$. Thus, for small $|\epsilon|$ and $\underline{s}$, the inequality $\lambda > z'$ will hold, thus forcing the conclusion that $s_1 \equiv 0$.

Here are some immediate corollaries to the last two lemmas:

Lemma 5.16. *Assume that the map q_0 from Lemma 5.12 is transversal to the varieties $\mathcal{D}$ and $\mathcal{D}'$ of Lemma 5.13. Then:*

1. *$(t, C) \in \mathcal{X}$ is a weak limit point only if there exists non-zero $\iota \in H^1(C; \mathbb{Z}/2)$ such that the kernel of D_ι on C is non-trivial.*

2. *The set of weak limit points in $\mathcal{X}$ is discrete and disjoint from the set of critical values of π.*

3. *A sequence $\{(t_k, C_k)\} \in \mathcal{X}$ which has $(t, C) \in \mathcal{X}_{\gamma,m}$ as its weak limit point must have all but finitely many members in $\mathcal{X}_{\gamma,2m}$.*

Proof of Lemma 5.16.

The lemma follows from the previous two lemmas with one extra observation: If $\underline{s}$ is in the kernel of D' on C', then the push forward of $\underline{s}$ gives an element s in the kernel of D acting on sections of $V_\rho \otimes N$, where ρ is defined from the covering map f. Conversely, a section of

$V_\rho \otimes N$ in the kernel of D defines an element $\underline{s}$ in the kernel of D' on some covering torus C' of C.

To be more explicit, remark that a covering map $f : C' \longrightarrow C$ of degree n defines a representation ρ of $\pi_1(C)$ into P_n as follows: Take any point in C. Then, label the inverse image as $\{1, \dots, n\}$. The group of deck transformations (a quotient group of $\pi_1(C)$) permutes this set and thus defines ρ. Conversely, a representation ρ of $\pi_1(C)$ into P_n defines an n-fold covering, $f : C' \longrightarrow C$, where C' is a complex torus, and f is a holomorphic covering map.

To see how sections of V_ρ on C correspond to functions $\underline{s}$ on C', note that the torus C' has the form $\mathbb{C}/H'$ where $H' \subset H$ is a sublattice. Thus, C' is tiled by copies of the fundamental domain of C. Label one of the fundamental domains of C in C' as $C[1]$. Then, the other copies have a unique labeling as $C[2], \dots, C[n]$ so that the action via ρ of $\mathbb{Z}\oplus\mathbb{Z}$ on $\{1, \dots, n\}$ describes how the group of deck transformations acts on the set $\{C[k]\}$. Now, let s be a section of V_ρ on C. Then s defines a complex valued function, $\underline{s}$, which is obtained as follows: The section s pulls up to $\mathbb{C}$ as an n-tuple of complex numbers, $(s_1, \dots, s_n)$. With this understood, define $\underline{s}$ by requiring that its restriction to the fundamental domain $C[k]$ equal s_k. Conversely, a complex valued function $\underline{s}$ on C' defines a section $s = (s_1, \dots, s_n)$ of V_ρ by setting $s_k = \underline{s}|_{C[k]}$.

Part 3. To continue the analysis of (5.37), consider now the case where $(t, C) \in \mathcal{X}$ has kernel(D_ι) non-trivial for some non-trivial class ι in $H^1(C; \mathbb{Z}/2)$. One can assume that this is the case for only one such ι, and that the kernel of D for C is trivial.

To begin, remark the group of deck transformations is isomorphic to $\mathbb{Z}/2$. With this undertood, the sections of f^*N decompose as direct sums $s_0 + s_1$, where s_0 is invariant under the non-trivial deck transformation, and s_1 changes sign under this deck transformation. As noted earlier, s_0 is the f-pull back of a section of N. Now, s_1 is the f-pull-back of a section of $\epsilon_\iota \otimes N$, where ϵ_ι is the real line bundle that is parameterized by the class ι. Thus, to say that D' has a non-deck invariant kernel is to say that the kernel of D_ι on C is non-trivial.

It is a straightforward task to set up a Banach space contraction mapping argument (this is the implicit function theorem in disguise) which finds a unique, small solution s_0 to (5.37.1) when $|\epsilon|$ and $|s_1|$ are small. The precise choice for the Banach space is immaterial save that it should control the sup norm of $\underline{s}$ and its covariant derivative. For example, one can use a Hölder space of sections which controls the Hölder norm of, at minimum, the first derivative of the section. (A similar contraction mapping construction is used in the proof of Lemmas 5.4 and 5.5 in [11] to which the reader is referred.) In any event, the application to (5.37.1) of a contraction mapping argument provides a unique small in norm s_0 which depends smoothly on ϵ and s_1 when the latter are small.

The cokernel of D' obstructs solvability of (5.37.2). With this understood, one proceeds by fixing non-zero $s \in \text{kernel}(D')$ and writing $s_1 = \alpha \cdot s + s_2$, where α is a real number, and s_2 is L^2 orthogonal to the kernel of D'. One can then set up a contraction mapping argument (as in the previous step) to find s_2 as a function of the pair (ϵ, α) when the latter are close to 0. Here, s_2 is found by solving the projection of (5.37.2) onto the orthogonal compliment of the cokernel of D'. The contraction mapping theorem asserts that when (ϵ, α) are close to zero, then there is a unique small solution $s_2 \equiv s_2(\epsilon, \alpha)$ to this projected equation. Furthermore, s_2 is a smooth function of its arguments.

Choose an element s^* in the cokernel of D'. With s_1 and s_2 now understood to be functions of (ϵ, α), the projection of (5.37.2) onto cokernel(D') defines a function g, on a neighborhood of the origin in $\mathbb{R}^2$, of the form

$$g(\epsilon, \alpha) = \alpha \cdot (r_1 \cdot \epsilon - h(\epsilon, \alpha)). \tag{5.39}$$

Part 4. The lemmas in this part describe the function g.

Lemma 5.17. *The function g in (5.39) has the following properties:*

1. *There is an embedding $g^{-1}(0) \cap \{\alpha > 0\}$ onto an open set $\mathcal{V} \subset \mathcal{X}$ with the property that (t, C) is not a weak limit point of any sequence in $\mathcal{X} - \mathcal{V}$. Also, this embedding composes with the map π to send (ϵ, α) to $t + \epsilon$.*
2. *$r_1 \neq 0$, and $h(0, \alpha)$ is not identically zero, but $h(0, 0) = 0$.*
3. *$h(\epsilon, -\alpha) = h(\epsilon, \alpha)$.*

Proof of Lemma 5.17. The embedding of $g^{-1}(0) \cap \{\alpha > 0\}$ onto an open set $\mathcal{V} \subset \mathcal{X}$ is given by associating to a pair (ϵ, α) the time $t + \epsilon$ and the embedded torus in X which coincides with the image in U of the section $\underline{s} = s_0 + \alpha s + s_2$. In this regard, note that the embeddings that are defined in this way using $\alpha < 0$ agree with those that are defined using $-\alpha > 0$. The reason is that $g(\epsilon, -\alpha)$ is equal to $-g(\epsilon, \alpha)$ since (5.37.2) is the projection onto the subspace which changes sign under the action of the non-trivial deck transformation. This means that $g^{-1}(0)$ is symmetric under changing α to $-\alpha$. Although this change effects the section $\underline{s}$, the effect is equivalent to that which is imposed by composing $\underline{s}$ with a deck transformation of f. Thus, the images of $\underline{s}$ as defined by α and $-\alpha$ agree. (The fact that g changes sign as α goes to $-\alpha$ explains Assertion 3.)

Lemmas 5.14-5.16 imply that $\mathcal{X}-\mathcal{V}$ contains no sequences with weak limit equal to (t, C). (This uses two facts: First, the small solution s_0 to (5.37.1) is uniquely defined by (ϵ, s_1) when $|\epsilon|$ and the C^0 norm of s_1 are small. Second, the small solution s_2 to the projected version of (5.37.2) is uniquely defined by (ϵ, α) when the latter are close to zero.)

As for the second assertion of Lemma 5.17, remark that $r_1 \neq 0$ is guaranteed by the fact that γ is transversal at (t, C) to the variety $\mathcal{D}$. This is proved by an argument which mimics that in Step 4 of the proof of Lemma 3.1. Indeed, the number r_1 is obtained by first extending D' to be a function of ϵ by considering the operator as defined by points $(t+\epsilon, C_\epsilon)$ in a neighborhood of (t, C) in $\mathcal{X}$. Then, r_1 is equal to the integral over C' of the ϵ-derivative at $\epsilon = 0$ of the function $s^*D's$. Meanwhile, an argument which is exactly analogous to that in Step 4 of Lemma 3.1 identifies a non-zero multiple of this same r_1 with the pull-back to $\mathcal{X}$ of the derivative at (t, C) of a function on $\mathcal{G}$, whose zero set (locally) defines the codimension 1 part of $\mathcal{D}$.

As for $h(0, \cdot)$, were it to vanish on an open set, then, according to Assertion 1, a line segment where $\epsilon = 0$ would lie in $\mathcal{V}$. However, the latter is precluded by the fact that π has non-degenerate critical points. The fact that $h(0,0) = 0$ is a direct consequence of its definition.

There is one last fact to establish here, and that concerns the function h in (5.39) for generic γ.

Lemma 5.18. *The conclusions of Lemma 5.8 can be amended to include the following: At each weak limit point $(t, C) \in \mathcal{X}$, the function h in (5.39) has the additional property that the Taylor's series of $h(0, \cdot)$ at $\alpha = 0$ is non-trivial. In fact, one can assume that this series starts with $h(0, \alpha) = r_2 \cdot \alpha^2 + \dots$, where $r_2 \neq 0$.*

Proof of Lemma 5.18. The fact that $h(0, \cdot)$ has a non-vanishing Taylor's expansion at zero is automatic in the case where X and the almost complex structure $\gamma(t)$ are real analytic. The general case can be had via a Sard-Smale argument along the lines that gave Assertion 3 of Proposition 5.2. The details are omitted.

End the digression.

Proof of Assertions 3-6 of Lemma 5.8.

To prove Assertion 3, remark first that Lemmas 5.14-5.17 imply that the set of weak limit points of $\mathcal{X}$ coincides with the set of points which are sent by Lemma 5.12's map q_0 into the part of the variety $\mathcal{D}$ of Lemma 5.13 where one of the operators from $\{D_\iota : \iota \neq 0\}$ has non-trivial kernel. Then Assertion 3 follows from the assumed transversality of q_0 to $\mathcal{D}$.

To prove Assertion 4, note that the fact that the critical points of π are non-degenerate implies that there is but a finite set of critical points which lie in the compliment of the union of all sets $\mathcal{V}$ from Lemma 5.17. There is one such $\mathcal{V}$ for each weak limit point in $\mathcal{X}$. Now fix attention to one of the sets $\mathcal{V}$. Since r_1 in (5.39) is non-zero, the coordinate α defines a smooth parameter on the set $\mathcal{V}$ (where α is close to zero.) Then, (5.39) and the implicit function theorem give ϵ as a function of α where α is small. According to Lemma 5.18, the function $h(0,\cdot)$ has a non-trivial Taylor's expansion at $\alpha = 0$ and thus so does ϵ since its expansion starts with the first term of that for $r_1^{-1} \cdot h(0,\alpha)$. Since ϵ has a non-trivial Taylor's expansion at $\alpha = 0$, there are only finitely many critical points of π on any given set $\mathcal{V}$.

With the preceding understood, the proof of Assertion 4 of Lemma 5.8 reduces to that for the assertion that the number of weak limit points in $\mathcal{X}$ is finite. In this regard, note that this number is locally finite because of the assumed transversality of q_0. To make further progress, consider the following induction proof: Observe first that the number of weak limit points in $\mathcal{X}_{\gamma,1}$ is finite since the latter is compact. Now, suppose that this number is finite in $\mathcal{X}_{\gamma,k}$ for all $k < m$ where $m \geq 1$ is assumed. If there is an infinite number of weak limit points in $\mathcal{X}_{\gamma,m}$, then there would be an infinite number in some $\mathcal{V}$ associated to a weak limit point (t, C) in $\mathcal{X}_{\gamma,m/2}$. In this case, one could find a sequence $\{(t_k, C_k)\} \in \mathcal{X}_{\gamma,2m}$ which had (t, C) as a weak limit point. This last possibility is precluded by Assertion 3 of Lemma 5.16.

The fifth assertion of Lemma 5.8 is implied by Lemma 5.3 together with the afore-mentioned fact that every point $(t, C) \in \mathcal{X}$ which is mapped by q_0 to $\mathcal{D} \cup \mathcal{D}'$ is either a critical point of π or else a weak limit point.

The final assertion is proved in Lemma 5.14.

Proof of Lemma 5.10. Assertions 1-3 follow directly from Lemmas 5.17 and 5.18. Assertion 4 follows since the weak limit points are disjoint from the set of critical points. Assertions 5 and 6 follow since the map q_0 in Lemma 5.12 can be assumed to be transversal to the subvarieties

$\mathcal{D}$ and $\mathcal{D}'$ of Lemma 5.13.

g) The proof of Lemma 5.11.

Here is the strategy: The operator D for the torus which is paremeterized by $\tau(t+\delta)$ is, for δ near zero, a perturbation of the analogous operator which is associated to the torus $C'(\delta)$ which double covers the torus that is parameterized by $\lambda(t+\delta)$. With this fact understood, perturbation theory computes the sign of the determinant of D for the former torus in terms of that for the latter. Meanwhile, the sign of the determinant of D for the latter can be computed in terms of the sign of the determinants of D and D_ι for the torus which is parameterized by $\lambda(t+\delta)$.

To prove Assertion 1, a straightforward application of perturbation theory finds that

$$\text{(5.40)} \qquad \operatorname{sign}(\det(D))|_{\operatorname{image}(\tau)} = -\operatorname{sign}(\det(D))|_{C'(\delta)}.$$

In fact, the relative signs here are the same as those of the differential (with ϵ fixed) of the function $r_1 \cdot \delta \cdot \alpha - h(\delta, \alpha)$ at $\alpha = 0$ and at its first small zero.

To compute the sign of $\det(D)$ for $C'(\delta)$, remember that this torus double covers the torus $C(\delta)$ from $\lambda(t+\delta)$. As remarked previously, the space of functions on $C'(\delta)$ decomposes as the direct sum of the deck-invariant functions and the functions which change sign under the action of the non-trivial deck transformatioon. The deck invariant functions are the f-pull-backs of functions on $C(\delta)$. A function which changes sign under the non-trivial deck transformation is the pull-back to $C'(\delta)$ of a section over $C(\delta)$ of the twisted line bundle ϵ_ι. Now, D on $C'(\delta)$ respects this decomposition. Thus,

$$\text{(5.41)} \qquad \operatorname{sign}(\det(D))|_{C'(\delta)} = \operatorname{sign}(\det(D))|_{C(\delta)} \operatorname{sign}(\det(D_\iota))|_{C(\delta)}.$$

These last two equations imply the first assertion of the lemma.

To prove the second assertion, label the non-trivial classes in $H^1(C; \mathbb{Z}/2)$ as ι, ι_1 and ι_2. Then $\iota' = f^*\iota_1 = f^*\iota_2$. With this last point understood, note that the non-trivial deck transformation pulls back ι' to itself, but even so, there is no natural lift of this deck transformation to the line bundle ϵ'_ι. Rather, there are two equally natural lifts, differing by multiplication by -1. With the choice of a lift, the space of sections over C' of ϵ'_ι decomposes as a direct sum of those sections, which are deck invariant, and those which change sign under the non-trivial deck transformation. After possibly renaming $\iota_{1,2}$, the former are identified with the sections which are pull-backs from C of sections of ι_1, and the latter are the pull-backs from C of sections of ι_2. Furthermore, the operator D'_ι on C' respects this splitting. Thus, D'_ι on $C'(\delta)$ has trivial kernel if and only if both D_{ι_1} and D_{ι_2} on $C(\delta)$ do, and if so, then, the sign of $\det(D'_\iota)$ is the product of the signs of $\det(D_{\iota_1})$ and $\det(D_{\iota_2})$. The assumption about the triviality of the kernel of $D_{\iota_{1,2}}$ is valid for δ near zero because of the assumed transversality of the map q_0. Finally, with the sign of $\det(D'_\iota)$ determined, perturbation theory insures that its sign is the same as that for the corresponding operator on the torus which is parameterized by $\tau(t+\delta)$.

To prove the third assertion of Lemma 5.11, consider first the operator D'_ι on C'. Any element in the kernel of this operator pushes forward to C as an element, s, in the kernel of the operator D on some $V_\rho \otimes N$, where ρ is a representation of $\pi_1(C)$ into P_4. More to the point, the subspace $O_s \subset \text{kernel}(D)$ will have dimension at least two. Thus, the assumption that q_0 in Lemma 5.12 is transversal to $\mathcal{D}'$ in Lemma 5.13 prevents such a kernel from appearing. Furthermore, the fact that $\mathcal{D}'$ has only strata with codimension two or more implies that $\text{sign}(\det(D'_\iota)) = 1$.

Finally, a standard perturbation theory argument shows that the sign for $\det(D'_\iota)$ on C' agrees with that for the analogous operator for the torus which is parameterized by $\tau(t+\delta)$ as long as δ is close to zero.

h) Proof of Lemma 5.13.

The first assertion of the lemma is a restatement of the results in Lemma 3.1, so only the second assertion need be considered here. The proof of the second assertion is accomplished in eight steps.

Step 1. To begin, write the complex torus C as $\mathbb{C}/H$. Now let P_n denote the permutation group on n letters, and let $\rho : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow P_n$ be a representation and write $V_\rho \longrightarrow C$ as $V_\rho \equiv \mathbb{C} \times_\rho \mathbb{C}^n$. Here, $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{C}$ so that the pair (n_1, n_2) translates a point in $\mathbb{C}$ by $n_1 + n_2\tau \in H$. In all of the following, assume that the representation ρ acts transitively on the set $\{1, \dots, n\}$.

This first step considers the possibilities for the dimension of the kernel of D on the space of sections of $V_\rho \otimes \mathbb{C}$.

Lemma 5.19. *If the ρ-action of $\mathbb{Z} \oplus \mathbb{Z}$ on $\{1, \dots, n\}$ has precisely one orbit, then the kernel of D on the space of sections of V_ρ has dimension 2 or less.*

Proof of Lemma 5.19. For the purposes of the proof, remember that the representation ρ also defines a finite, holomorphic covering, $f : C' \longrightarrow C$, where C' is a complex torus, and ρ is a holomorphic covering map. This covering has the property that f^*V_ρ is holomorphically isomorphic to the trivial complex n-plane bundle. More to the point, the sheaf of sections of V_ρ is isomorphic to the push-forward by f of the sheaf of sections of the trivial bundle over C. Thus, a function $\underline{s}$ on C' corresponds to a section s of V_ρ, and vice-versa. Furthermore, the section s is in the kernel of D if and only if the function $\underline{s}$ on C' is annihilated by the operator D' on C', defined as in (2.7) with ν and μ given by the f-pull-backs of their analogs defining the operator D on C.

According to Lemma 5.1, the kernel of D' is at most 2 dimensional, and thus, so is the kernel of D on the space of sections of $V_\rho \otimes \mathbb{C}$.

Step 2. Now introduce the set $\mathcal{D}_\rho \subset \mathcal{G}$ of triples $(\tau, (\mu, \nu))$ for which the operator D has a section s of $V_\rho \otimes \mathbb{C}$ in its kernel with $\dim(O_s) = 2$. This step constructs a local model for a neighborhood in $\mathcal{D}_\rho$ of a given pair $\xi \equiv (\tau, (\nu, \mu))$.

To begin, suppose that $\xi^0 \equiv (\tau + \tau^0, (\nu + \nu^0, \mu + \mu^0)) \in \mathcal{D}_\rho$ is close to ξ. To avoid confusion, agree to use D to denote the operator in (2.7) as defined by ξ, and use D^0 to denote the analogous operator as defined by ξ^0. The fact that ξ lies in $\mathcal{D}_\rho$ means that there is a set (s, t) of linearly independent elements in the kernel of D. There is a similar set $(s + s^0, t + t^0)$ for the kernel of D^0. If $|\tau^0|$ and the C^0 norms of ν^0 and μ^0 are small, then (s^0, t^0) will be small and can be chosen to be L^2-orthogonal to the span of (s, t). Then, s^0 is completely characterized by this last condition, and by the condition that it obeys a certain equation of the form

$$(5.42) \qquad Ds_1 + h(\tau^0) \cdot \nabla(s + s^0) + \nu^0(s + s^0) + \mu^0(\bar{s} + \bar{s}^0) = 0.$$

Here, h is a certain section of $\mathrm{Hom}(T^*C; T^{0,1}C \otimes N)$ which depends analytically on τ^0 and vanishes when $\tau^0 = 0$. Also, t^0 is completely characterized by the condition that it be L^2-orthogonal to the kernel of D and that it solve the analog of (5.42) where t replaces s and where t^0 replaces s^0 everywhere.

Now, the operator D is not invertible; and as its index is zero, its cokernel has dimension 2. This cokernel is isomorphic in a natural way to the kernel of the L^2-adjoint of D, an operator which will be denoted by D^*. And, the operator D can be inverted only on sections of V_ρ which are L^2-orthogonal to the kernel of the D^*. With the preceding understood, remark that without constraints on $(\tau^0, (\nu^0, \mu^0))$, one can, at best, hope to solve only

$$Ds^0 + \prod(h(\tau^0) \cdot (\nabla(s + s^0) + \nu^0(s + s^0) + \mu^0(\bar{s} + \bar{s}^0)) = 0, \tag{5.43}$$

where $\prod$ is the L^2-orthogonal projection onto the compliment of the kernel of D^*. For small τ^0 and small (ν^0, μ^0) (say, in the $C^{0,1/2}$ Hölder space topology), a fixed point construction finds a unique solution s^0 to (5.43), which is L^2-orthogonal to the kernel of D, and has $C^{1,1/2}$ Hölder norm bounded by a uniform constant times the sum of $|\tau^0|$ and the $C^{0,1/2}$ Hölder norm of (ν^0, μ^0). Furthermore, s^0 has real analytic dependence on the pair (ν^0, μ^0). (Here, one can use the C^∞-Frechet topology because the operator D is elliptic.) The analogous assertion holds for t^0.

Now, let $\prod^c \equiv 1 - \prod$ as an operator on the space of sections of V_ρ. This is the L^2-orthogonal projection onto the kernel of D^*. It follows from (5.42) and (5.43) that the intersection of $\mathcal{D}_\rho$ with an appropriate $C^{0,1/2}$ neighborhood of ξ is homeomorphic to the subspace of a neighborhood of the origin in $\mathbb{C} \times_2 C^\infty(C; \mathbb{C})$ which consists of triples $(\tau^0, (\nu^0, \mu^0))$ satisfying the following two equations:

$$\begin{aligned} &1. \textstyle\prod^c(h(\tau^0) \cdot \nabla(s + s^0) + \nu^0(s + s^0) + \mu^0(\bar{s} + \bar{s}^0)) = 0. \\ &2. \textstyle\prod^c(h(\tau^0) \cdot \nabla(t + t^0) + \nu^0(t + t^0) + \mu^0(\bar{t} + \bar{t}^0)) = 0, \end{aligned} \tag{5.44}$$

where the pair (s^0, t^0) should be thought of as real analytic functions which are defined on a small ball about the origin in $\mathbb{C} \times_2 C^\infty(C; \mathbb{C})$. To summarize: Equations (5.44) define a real analytic map, F, from a ball in $\mathbb{C} \times C^\infty(C; \mathbb{C})$ into $\times_2 \operatorname{kernel}(D^*)$, and $F^{-1}(0,0)$ is homeomorphic to a neighborhood of ξ in $\mathcal{D}_\rho$.

The analysis of $F^{-1}(0,0)$ proceeds by analyzing the differential of F at $(0,0)$. This differential is a linear map from $\mathbb{C} \times_2 C^\infty(C; \mathbb{C})$ to $\times_2 \operatorname{Kernel}(D)$. To make this differential concrete, choose a linearly independent pair $(u, v) \in \operatorname{kernel}(D^*)$. Then, the differential of F at $(0,0)$ is the linear map which sends $(\tau^0, (\nu^0, \mu^0))$ to

$$F_*(\tau^0, (\nu^0, \mu^0)) \equiv \int_C \operatorname{Real}(f), \tag{5.45a}$$

where f is the following 2×2 complex valued matrix:

$$
\begin{aligned}
f \equiv & h(\tau^0) \sum_k \begin{pmatrix} \overline{u}_k \nabla s_k & \overline{u}_k \nabla t_k \\ \overline{v}_k \nabla s_k & \overline{v}_k \nabla t_k \end{pmatrix} \\
& + \nu^0 \sum_k \begin{pmatrix} \overline{u}_k s_k & \overline{u}_k t_k \\ \overline{v}_k s_k & \overline{v}_k t_k \end{pmatrix} + \mu^0 \sum_k \begin{pmatrix} \overline{u}_k \overline{s}_k & \overline{u}_k \overline{t}_k \\ \overline{v}_k \overline{s}_k & \overline{v}_k \overline{t}_k \end{pmatrix}.
\end{aligned}
\tag{5.45b}
$$

Here, $(s_1, \ldots, s_n)$ are the components of the pull-back of s to $\mathbb{C}$, and the t_k, u_k, and v_k are defined analogously.

The analysis of (5.45a) requires a lengthy digression to consider various properties of the kernel and cokernel of D. This digression occupies the next five steps. The last step of the proof returns to the milieu of (5.45a) to finish the story.

Step 3. This step reviews some group theory which is necessary for the analysis of the kernels of D and D^*. To begin, suppose that there is but one orbit for the action via ρ of $\mathbb{Z} \oplus \mathbb{Z}$ on the set $\{1, \ldots, n\}$. Let T_1 denote $\rho(1,0)$ and let T_2 denote $\rho(0,1)$. Both T_1 and T_2 generate cyclic groups acting on $\{1, \ldots, n\}$ of orders n_1 and n_2, respectively. Assume, without loss of generality, that $n_1 \geq n_2$. There are two cases now to consider. In the first case, $n_1 = n$. In this case, $\{1, \ldots, n\}$ can be relabled so that $T_1 \cdot k = (k+1)_{\text{ mod }(n)}$. Here, $T_2 = T_1^p$ for some integer p in the set $\{0, \ldots, n-1\}$.

In the second case, $n_1 < n$. Here, n_1 must divide n; and then $n_2 \equiv n/n_1$. In this case, the set $\{1, \ldots, n\}$ can be relabled so that

$$
\begin{aligned}
T_1 \cdot (kn_1 + 1 + q) &= kn_1 + 1 + (q+1)_{\text{ mod }(n_1)}, \\
T_2 \cdot (kn_1 + 1 + q) &= (k+1)_{\text{ mod }(n_2)} \cdot n_1 + 1 + q,
\end{aligned}
\tag{5.46}
$$

where $k \in \{0, \ldots, n_2 - 1\}$ and $q \in \{0, \ldots, n_1 - 1\}$.

Step 4. Assume, as before, that ρ has one orbit in $\{1, \ldots, n\}$. When $s \in \text{kernel}(D)$, reintroduce $O_s \subset \text{kernel}(D)$, which is the representation space of $\mathbb{Z} \oplus \mathbb{Z}$ that is spanned by vectors of the form $\rho(a) \cdot s$ where $a \in \mathbb{Z} \oplus \mathbb{Z}$. The dimension constraint on the kernel of D implies that the dimension of O_s is either 1 or 2. In either case, one has

$$
\begin{aligned}
T_1 \cdot s &= \cos(\theta_1) s + \sin(\theta_1) t, \\
T_2 \cdot s &= \cos(\theta_2) s + \sin(\theta_2) t.
\end{aligned}
\tag{5.47}
$$

Here θ_1 and θ_2 take values in $[0, 2\pi)$, and $\sin(\theta_1)$ and $\sin(\theta_2)$ are not both zero in the two dimensional case. Of course, t is superfluous when $\dim(O_s) = 1$; when $\dim(O_s) = 2$, then t is some element in the kernel of D, which is linearly independent from s. To make t more precise, it proves useful to pull s up to $\mathbb{C}$ and write $s = (s_1, \ldots, s_n)$. Then, $s_1(0)$ cannot vanish because, as remarked in Lemma 5.1, $\underline{s}$ is nowhere vanishing on C_ρ. With this understood, a metric on kernel(D) is defined by taking the inner product between elements s and t to be

$$\langle s, t \rangle \equiv \sum_k \mathrm{Real}(\bar{s}_k(0) t_k(0)). \tag{5.48}$$

As this metric is invariant under the action of P_n on $\mathbb{C}_n$, one can choose t in (5.47) to have the same norm as s and be orthogonal to s under the metric in (5.48).

Note that (5.47) plus the vanishing of (5.48) does not uniquely define the triple (θ_1, θ_2, t). There are two choices; given the first, the second is $(2\pi - \theta_1, 2\pi - \theta_2, -t)$. This sign ambiguity is fixed with the choice of an orientation for O_s. With the preceding understood, the convention here will be to orient O_s so that the restriction of $(\underline{s}, \underline{t})$ to any point in C_ρ gives the correct orientation for $\mathbb{C}$. Lemma 5.1 insures that $\underline{s}$ and $\underline{t}$ are everywhere linearly independent.

For the next remark, note that t must transform under ρ as

$$\begin{aligned} T_1 \cdot t &= \cos(\theta_1) t - \sin(\theta_1) s, \\ T_2 \cdot t &= \cos(\theta_2) t + \sin(\theta_2) s. \end{aligned} \tag{5.49}$$

Indeed, (5.49) follows from (5.47) with the observation that both T_1 and T_2 must act as orientation preserving isometries of O_s. To see that such is the case for T_1 (for example), note first that $\underline{T_1 \cdot s}$ is obtained from $\underline{s}$ by translating $\underline{s}$ by an appropriate deck transformation of the covering $f : C_\rho \longrightarrow C$. And, $\underline{T_1 \cdot t}$ is obtained from $\underline{t}$ in a similar fashion. Since $\underline{s}$ and $\underline{t}$ are everywhere linearly independent (by Lemma 5.1), and C_ρ is connected, it follows that the restrictions of the ordered pairs $(\underline{s}, \underline{t})$ and $(\underline{T_1 \cdot s}, \underline{T_1 \cdot t})$ to any given point in C_ρ must define the same orientation on $\mathbb{C}$.

As a final remark in this step, note that (5.48) and (5.49) imply that $\zeta_1 \equiv \exp(i\theta_1)$ obeys $\zeta_1^{n_1} = 1$ with n_1 being the order of T_1. Likewise, $\zeta_2 \equiv \exp(i\theta_2)$ obeys $\zeta_2^{n_2} = 1$ where n_2 is the order of T_2.

Step 5. Fix $s \in \text{kernel}(D)$ and assume that O_s is 2-dimensional. Write the pull-back to $\mathbb{C}$ of s as $(s_1, \dots, s_n)$, and similarly write the pull-back of t as $(t_1, \dots, t_n)$. In this case, each of s_k and t_k is determined by s_1 and t_1. To be precise,

$$\begin{aligned} s_{kn_1+q+1} &= \cos(q\theta_1 + k\theta_2)s_1 - \sin(q\theta_1 + k\theta_2)t_1, \\ t_{kn_1+q+1} &= \sin(q\theta_1 + k\theta_2)s_1 + \cos(q\theta_1 + k\theta_2)t_1, \end{aligned} \tag{5.50}$$

where $q \in \{0, \dots, n_1 - 1\}$ and $k \in \{0, \dots n_2 - 1\}$. In the case where $n_1 = n$, the preceding formula holds with $k \equiv 0$.

Step 6. Make a similar analysis for the kernel of the L^2- adjoint of the operator D in the case where $\dim(O_s) = 2$. Note that this L^2 adjoint D^* of D sends a section w of V_ρ to

$$D^* w \equiv -\partial w + \overline{\nu} w + \mu \overline{w}. \tag{5.51}$$

The kernel of D^* is spanned by $u \equiv (u_1, \dots, u_n)$ and a similar $v \equiv (v_1, \dots, v_n)$. As D has zero index, the kernel of D^* is also a 2 dimensional vector space, L_s.

Lemma 5.20. *Basis elements $u \equiv (u_1, \dots, u_n)$ and $v \equiv (v_1, \dots, v_n)$ for L_s can be chosen so that either*

$$\sum_k \begin{pmatrix} \overline{u}_k(0)s_k(0) & \overline{u}_k(0)t_k(0) \\ \overline{v}_k(0)s_k(0) & \overline{v}_k(0)t_k(0) \end{pmatrix} = n \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \tag{5.52a}$$

or else

$$\sum_k \begin{pmatrix} u_k(0)s_k(0) & u_k(0)t_k(0) \\ v_k(0)s_k(0) & v_k(0)t_k(0) \end{pmatrix} = n \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{5.52b}$$

Furthermore, in the first case, (5.52b) is zero, and in the second case, (5.52a) is zero.

Proof of Lemma 5.20. There are two parts to the proof of this lemma. The first part starts by observing that there is a self-evident version of Lemma 5.1 for D^* with the following implications: First, a basis (u, v) for L_s can be chosen to obey the normalization $\langle u, u\rangle = \langle v, v\rangle \neq 0$ and $\langle u, v\rangle = 0$ with $\langle , \rangle$ as in (5.48). Second, the group $\mathbb{Z} \oplus \mathbb{Z}$ acts on the 2-dimensional space spanned by the pair (u, v), and (5.47), (5.49) can be assumed obeyed with the replacement of s by u and t by v. Here, the angles (θ_1, θ_2) can be assumed to be the same as those in the

(s,t) version of (5.47) and (5.49). This follows from the constancy of the group invariant index of the operator D on $V_\rho \otimes \mathbb{C}$ under deformations which move ν and μ to zero. (See Lemma 3.1.)

Note that there is some cost to making the angles in the (u,v) version the same as those in the (s,t) version: There is no guarantee that the orientation of L_s as defined by (u,v) gives, upon restriction of $(\underline{u},\underline{v})$ to a point in C_ρ, the correct orientation for $\mathbb{C}$. In fact, if this orientation is correct, then (5.52a) will be seen to hold, and if said orientation is incorrect, then (5.52b) will be seen to hold.

In any event, given that there is a (u,v) version of (5.47) and (5.49), then one can conclude that the components $\{u_k\}$ and $\{v_k\}$ obey (5.50) with the replacement of s_k by u_k and likewise t_k by v_k.

The second part of the proof remarks that $\Sigma_k \overline{u}_k s_k$, and similar sums can be computed in terms of s_1 and u_1 with the help of (5.50) and its (u,v) analog. In particular, a straightforward calculation finds that

$$\sum_k \begin{pmatrix} \overline{u}_k s_k & \overline{u}_k t_k \\ \overline{v}_k s_k & \overline{v}_k t_k \end{pmatrix} = \frac{n}{2} \begin{pmatrix} \overline{u}_1 s_1 + \overline{v}_1 t_1 & \overline{u}_1 t_1 - \overline{v}_1 s_1 \\ -\overline{u}_1 t_1 + \overline{v}_1 s_1 & \overline{u}_1 s_1 + \overline{v}_1 t_1 \end{pmatrix} \tag{5.53a}$$

and

$$\sum_k \begin{pmatrix} u_k s_k & u_k t_k \\ v_k s_k & v_k t_k \end{pmatrix} = \frac{n}{2} \begin{pmatrix} u_1 s_1 + v_1 t_1 & u_1 t_1 - v_1 s_1 \\ -u_1 t_1 + v_1 s_1 & u_1 s_1 + v_1 t_1 \end{pmatrix}. \tag{5.53b}$$

By the way, this last equation holds for any 2 pairs (s,t) and (u,v) of sections of V_ρ as long as each pair satisfies (5.50). In particular, using s for u and t for v in (5.53a), this last equation implies that the assumed orthogonality with respect to (5.48) for s and t and the choice of orientation for O_s require that $t_1(0) = i \cdot s_1(0)$. Likewise, the correct orthogonality for (u,v) with respect to (5.48) requires $v_1(0) = \pm i \cdot u_1(0)$. In principle, $u_1(0)$ can be arbitrary, but to obtain Lemma 5.20, the choice $u_1(0) = \frac{1}{\overline{s}_1(0)}$ must be made when $v_1(0) = iu_i(0)$. With this choice, (5.53) implies Lemma 5.20. Indeed, in the case where $v_1(0) = i \cdot u_1(0)$ and $u_1(0) = \frac{1}{\overline{s}_1(0)}$, then (5.52a) holds and the left-hand side of (5.52b) is zero. In the case where $v_1(0) = -i \cdot u_1(0)$, one should choose $u_1(0) = \frac{1}{s_1(0)}$, and then the left-hand side of (5.52a) is zero and (5.52b) holds.

Step 7. This step is not really needed for the proof of Lemma 5.13 but it is included for completeness. It considers the case where $s \in$

kernel(D) has dimension one O_s. Now, $T_1 \cdot s = \epsilon_1 s$ and also $T_2 \cdot s = \epsilon_2 s$, with $\epsilon_{1,2} = \pm 1$. That is, the action of ρ on O_s factors through $\mathbb{Z}/2$.

In this case, the pull-back to $\mathbb{C}$ of s has the form $(s_1, \ldots, s_n)$, and the preceding behavior under the action of ρ implies that $s_2 = \epsilon_1 s_1$ and, in general, $s_k = \pm s_1$ where the choice of sign is determined by ϵ_1 and ϵ_2 via (5.46). Note in particular that when n is odd, the representation is the trivial one.

Now, suppose that there exists a second element $t \in$ kernel(D) which is linearly independent of s. Because of Lemma 5.1 the dimension of O_t is one also. Furthermore, consider

Lemma 5.21. *In the circumstances above, O_s is isomorphic to O_t as a representaton of $\mathbb{Z} \oplus \mathbb{Z}$. In addition, the cokernel of D is also a direct sum of two 1-dimensional representations of $\mathbb{Z} \oplus \mathbb{Z}$, with each being isomorphic to O_s. Furthermore, the elements (s, t) in the kernel and (u, v) in the kernel of D^* can be chosen so that the elements of each pair form an oriented, orthonormal basis for* kernel(D) *and* cokernel(D^*), *respectively, with respect to the inner product in (5.48). Finally, (5.53a) can be assumed to hold.*

Proof of Lemma 5.21. To prove the first assertion, one must show that T_1 acts with the same sign on s and t, and likewise for T_2. To see that such is the case, remember that $(\underline{s}, \underline{t})$ are linearly independent everywhere on C' and so define a consistent orientation for $\mathbb{C}$ everywhere on C'. This is impossible when one of (s, t) changes sign with the application of T_1 (or T_2) and the other does not.

The second assertion of Lemma 5.21 follows from the first with the equivariant index theorem. The third assertion, that concerning the orthogonality of the pairs (s, t) and (u, v), can be proved by standard linear algebra arguments. The assertion about (5.53a) follows from the fact that each $s_k = \epsilon_k s_1, t_k = \epsilon_k t_1$ and similarly for u_k and v_k, where $\epsilon_k = \pm 1$ is the same for s_k, t_k, u_k and v_k.

Step 8. This step returns to (5.45) to study the differential of the map F. Assume here that O_s is two-dimensional. To begin, remark that the vanishing of F^* defines not 4 equations but only two. This is because the pairing between (s, t) and (u, v) which appears in (5.45b) is invariant under the action of $\mathbb{Z} \oplus \mathbb{Z}$. To put this in perspective, fix $(\tau^0, (\nu^0, \mu^0))$ in (5.44) and then consider this equation as a map from kernel(D) to kernel(D^*). In this guise, (5.44) is linear and equivariant under the action of $\mathbb{Z} \oplus \mathbb{Z}$. This follows from the fixed point construction of (s^0, t^0). Thus, the vanishing of the top row of the matrix F^* implies the vanishing of the bottom row too.

The point of the proceeding is that the restriction of F^* to the top row in (5.45a, b) defines a linear map (call it F_*^1) from $\times_2 C^\infty(C;\mathbb{C})$ to $\mathbb{R}^2$. This last map is surjective, a fact which follows directly from Lemma 5.20. The surjectivity of F_*^1 implies, with the help of the implicit function theorem, that $\mathcal{D}_\rho$ is locally a codimension-2 submanifold of $\mathbb{C} \times_2 C^\infty(C;\mathbb{C})$ near $(\tau, (\nu, \mu))$. (Remember here that F defines, for fixed (ν^0, μ^0), a $\mathbb{Z} \oplus \mathbb{Z}$ equivariant map from kernel(D) to kernel(D^*). Thus, if the top row of F vanishes for a given triple $(\tau^0, (\nu^0, \mu^0))$, then the bottom row will vanish too.)

Proof of Theorem 1.2. What follows is an outline of the proof. The discussion here is abbreviated because the argument is, for the most part, the same in its essentials as that which proved Theorem 1.1. Most aspects of the discussion also appear as parts of Sections 6 and 7 of [5].

To begin, fix a class $e \in H^2(X;\mathbb{Z})$ and then consider the definition of $\mathrm{GW}(e)(\cdot)$ in Theorem 1.2 with the following modification to the definition: Require that $\mathcal{H}(e, J, \Gamma, \Omega)$ consist of the, set of connected, pseudo-holomorphic submanifolds which contain all members of Ω, which intersect precisely once all members of Γ, and whose fundamental class is Poincaré dual to e. Count the elements in $\mathcal{H}$ by associating to each element a sign, with the latter defined (assuming $\Gamma \neq \emptyset$) as in (2.15). Denote the resulting number by $\mathrm{RW}(e)(J, \Gamma, \Omega)$.

Arguments such as those which proved Proposition 5.2 can be used to establish that $\mathrm{RW}(e)(J, \Gamma, \Omega)$ is well defined if the triple (J, Γ, Ω) are chosen from a suitably generic set. Indeed, some straightforward modifications to the arguments in the proof of Proposition 5.2 prove that there is a Baire set of triples (J, Γ, Ω) for which the following are true:

(5.54)

- The set $\mathcal{H}$ contains only finitely many elements.
- If (J', Ω', Γ') is sufficiently close to (J, Ω, Γ) in the Baire set, then the corresponding set $\mathcal{H}'$ has the same number of elements as does $\mathcal{H}$.
- When $C \in \mathcal{H}$, then the operator D has trivial cokernel, and the evaluation map H from kernel(D) to $(\oplus_{z \in \Omega'} N|_z) \oplus (\oplus_{1 \leq \alpha \leq 2p} 1_\alpha)$ is an isomorphism.

Note that under the assumption that $\Gamma \neq \emptyset$, considerations of multiply covered tori are irrelevant. For example, when Γ contains $2n$ elements, then a relevant "universal model" for the proof of (5.54) is a space which consists of Diff(Σ) orbits of data sets of the form $[(z),(y),(v),j,\varphi],J,(x),(u)]$, where

(5.55)

- $(z) = (z_1,\ldots,z_{d'-n'})$, $(y) = (y_1,\ldots,y_p)$ and also $(v) = (v_1,\ldots,v_{2n'})$ are sets of unordered points in Σ (here, $d' \geq d$ and $n' \geq n$),
- j is an almost complex structure on Σ which maps to some L_i from (5.5),
- φ is a smooth map from Σ into X such that $\varphi_*[\Sigma] = e$,
- J is an ω-compatible, almost complex structure on X,
- $(x) = (x_1,\ldots,x_{d'-n'})$ is a set of distinct points on X, and
- $(u) = (u_1,\ldots,u_{2n'})$ is such that $u_\alpha \in \gamma_{\alpha'}$.

And, this data is constrained so that the following hold:

(5.56)

1. $\varphi_* j = J\varphi_*$.
2. The differential of φ vanishes at each y_k.
3. $\varphi(z_k) = x_k$.
4. $\varphi(y_k) = u_k$.

5, φ is an embedding off of a finite set of points.

Meanwhile, the arguments from Chapter 7 of [5] can be readily adapted to the present situation to prove that $\mathrm{RW}(e)(J,\Gamma,\Omega)$ depends only on e and the classes of the elements of Γ in $H_1(X;\mathbb{Z})$.

The arguments which prove Proposition 5.2 establish that $\mathrm{RW}(e)(\cdot)$ takes the same value (J,Γ,Ω) and on (J',Γ',Ω') as long as each element in Γ is isotopic to the corresponding element in Γ'. However, the step from isotopy to homology is not complicated. In fact, one can modify Proposition 5.2's arguments directly by considering universal models as in (5.55) and (5.56) where one or more of the u_k are constrained to lie in some submanfold with boundary in X.

Given that $\mathrm{RW}(e)(\cdot)$ depends only on the homology classes of the elements of Γ, it follows automatically that $\mathrm{RW}(e)(\cdot)$ is in

$$\oplus_{p\geq 0}(\otimes_p(H_1(X;\mathbb{Z})/\,\mathrm{Torsion}).$$

Indeed, if $[\gamma]=[\gamma_1]+[\gamma_2]$, then $[\gamma]$ can be represented by the disjoint union $\gamma_1\cup\gamma_2$, of embedded submanifolds, where γ_1 represents $[\gamma_1]$, and γ_2 represents $[\gamma_2]$. And, in this case, the sum which defines $\mathrm{RW}(e)(J,\Gamma,\Omega)$ is identical to that which is obtained by adding those sums which define $\mathrm{RW}(e)(J,\Gamma_1,\Omega)$ and $\mathrm{RW}(e)(J,\Gamma_2,\Omega)$. It also follows from the definition of the sign in (2.15) that the multi-linear functional $\mathrm{RW}(e)(\cdot)$ is an antisymmetric functional. (By construction, $\mathrm{RW}(e)(\cdot)$ vanishes on $\Lambda^p(\cdot)$ when p is odd and when $p>2\cdot d(e)$.)

With the, invariant $\mathrm{RW}(e)(\cdot)$ understood, the proof of Theorem 1.2 follows by rewriting $\mathrm{GW}(e)(\cdot)$ as sums and products of the RW and Qu; the formula here is analogous to the formula for $\mathrm{Gr}(e)$ in (5.12) and Lemma 5.6.

6. A toroidal example

The purpose of this section is to provide some examples of symplectic manifolds which illustrate the necessity of at least some of the complications with multiply toroidal classes in the definition of $\mathrm{Gr}(e)$. In particular, the examples here show that there is no reasonable invariant to assign to a multiply toroidal class e that counts only pseudo-holomorphic submanifolds with fundamental class Poincaré dual to e.

To begin the story, fix a compact, oriented surface Σ. Let g denote the genus of Σ. Now, set

$$X\equiv S^1\times S^1\times\Sigma. \tag{6.1}$$

To define a symplectic form on X, introduce a standard coordinate, $t_1\in[0,1]$, for the leftmost circle in (6.1). Let $t_2\in[0,1]$ denote a similar coordinate for the middle factor of S^1 in (6.1). Fix a volume form w for Σ and fix a smooth, closed 1-form f on Σ with only non-degenerate zeros. With the preceding understood, set

$$\omega\equiv dt_1\wedge dt_2+w+dt_2\wedge f. \tag{6.2}$$

Note that any two symplectic forms which are described by (6.2) can be connected by a continuous path of symplectic forms.

Now, consider the following compatible almost complex structure: Define J so that J maps the $T\Sigma$ summand in TX to itself and so that the restriction of J on $T\Sigma$ is compatible with w. In addition, define

$$J \cdot \frac{\partial}{\partial t_1} = \frac{\partial}{\partial t_2} + w^{-1} \cdot f, \tag{6.3}$$

where, w^{-1} is the section of $\Lambda^2 T\Sigma$ which pairs with w to get 1.

Note that the canonical bundle of this X is isomorphic (as an oriented 2-plane bundle) to the pull-back via projection to Σ of the canonical bundle to Σ. With this understood, the cohomology class c can be written as $(2g-2) \cdot e_1$ where e_1 is the class in $H^2(X;\mathbb{Z})$ which is Poincaré dual to a copy of $S^1 \times S^1 \times \{\text{point}\}$. Since $e_1 \cdot e_1 = 0$, any connected, pseudo-holomorphic curve whose Poincaré dual is a multiple of e_1 will be a torus with trivial normal bundle.

The next task is to determine the pseudo-holomorphic submanifolds which are Poincaré dual to $n \cdot e_1$ when $n \geq 1$ is a positive integer. With n fixed, the on going assumption will be that f and ∇f have small norm.

The first lemma below describes the relevant pseudo-holomorphic submanifolds. The statement of the lemma requires the introduction of the sets $\text{zero}_+\{f\}$ and $\text{zero}_-\{f\}$ of zeros of f with, respectively, even and odd indices. (Locally, $f = dh$ for a function h, and the $\pm$ here can be interpreted as the sign of the determinant of the Hessian of h at the critical point.)

Lemma 6.1. *Fix $n \geq 1$. There exists $\epsilon > 0$ with the following significance: Let f be a closed 1-form with only non-degenerate zero points and with $|f| + |\nabla f| < \epsilon$ everywhere.*

1. *The connected, pseudo-holomorphic submanifolds are of the form $(S^1 \times S^1) \times p$, where p is a zero point of f.*
2. *These submanifolds are non-degenerate in the sense of Definition 2.1, and they are n-non-degenerate in the sense of Definition 4.1.*
3. *If $C = S^1 \times S^1 \times p$ with p a zero point of f, then the sign of* $\det(D)$ *(as in (2.13)) is $+1$ if $p \in \text{zero}_+(f)$, and -1 if $p \in \text{zero}_-(f)$.*
4. *If $C = S^1 \times S^1 \times p$ with p a zero point of f, and $\iota \in H^1(C;\mathbb{Z}/2)$ is non-trivial, then* $\text{sign}(\det(D_\iota)) = 1$.

5. J is from the Baire subset which is described in (5.15).

This lemma is proved below.

With the lemma understood, one can compute the sum (1.7) as follows: First of all, if C is a connected, pseudo-holomorphic submanifold which appears (with some multiplicity, m) as an element of some $\{(C_k, m_k)\}$ in $\mathcal{H}(e, J, \emptyset)$, then the Poincaré dual of $[C_k]$ must be a multiple of e_1. This follows because the pseudo-holomorphic submanifolds have locally positive intersection number, and thus must have positive intersection number with the torus $S^1 \times S^1 \times \{\text{zero of } f\}$. Hence, in this case, $\mathrm{Gr}(n \cdot e_1)$ equals $\mathrm{Qu}(e_1, n)$. The computation of the latter is facilitated by introducing the generating function

$$G[z] \equiv 1 + \sum_{n \geq 1} \mathrm{Qu}(e_1, n) z^n. \tag{6.4}$$

Then it follows from the definition of $\mathrm{Qu}(e_1, n)$ in (5.16) and the definition of $P_{\pm 0}$ in (3.2) that $G[z]$ can be rewritten as a product indexed by the zeros of f. Here, each $p \in \mathrm{zero}_+ f$ contributes a factor of P_{+0} (due to Assertions 3 and 4 of Lemma 6.1), and each $p \in \mathrm{zero}_-\{f\}$ contributes a factor of P_{-0}. Thus, if m denotes the size of $\mathrm{zero}_+(f)$, one has

$$\begin{aligned} G[z] &= (f_{+0})^{m+2}(f_{-0})^{2g+m} \\ &= (1-z)^{-m-2}(1-z)^{2g+m} = (1-z)^{2g-2}. \end{aligned} \tag{6.5}$$

It is amusing to compare (6.5) with a count which includes only pseudo-holomorphic submanifolds. For this purpose, note that when n is larger than the number of zero of f, then according to Lemma 6.1, there are no pseudo-holomorphic submanifolds in the class $n \cdot e_1$. Therefore, the pseudo-holomorphic submanifold count gives zero in this case. If n is no greater than the number of zeros of f, then, according to Lemma 6.1, the psuedo-holomorphic submanifolds of f in the class $n \cdot e_1$ consist of the submanifolds of the form $C \equiv S^1 \times S^1 \times \Lambda$, where Λ is a set of n distinct zeros of f. Use the rule in (2.13) to define the sign for the contribution of each such manifold. That is, when Λ is a set of j distinct points in $\mathrm{zero}_+\{f\}$ and k distinct points in $\mathrm{zero}_-\{f\}$, then the pseudo-holomorphic submanifold $(S^1 \times S^1) \times \Lambda$ contributes $(-1)^k$ to the sum for this hypothetical "invariant" for the class $(j+k) \cdot e_1$. With this contribution understood, it follows that the resulting generating function for the pseudo-holomorphic submanifold count is equal to

$$(1+z)^{m+2}(1-z)^{2g+m} = (1-z)^{2g-2}(1-z^2)^{m+2}. \tag{6.6}$$

This generating function has an unpleasant dependence on the number of zeros of f.

Here is another natural, but erroneous suggestion to repair (6.6): Add to the count multiple covers of $(S^1 \times S^1) \times p$ with the following weights: Count a q-fold cover of $(S^1 \times S^1) \times p$ as $+1$ when $p \in \text{zero}_+\{f\}$, and count it as $(-1)^q$ when $p \in \text{zero}_-\{f\}$. This count produces the generating functional

$$(1 - z)^{-m-2}(1 + z)^{-2g-m} = (1 + z)^{-2g+2}(1 - z^2)^{-m-2}, \tag{6.7}$$

which also has an unpleasant dependence on the number of zeros.

The proof of Theorem 1.1 should make clear the fact that the contribution to the generating functional from a point in $\text{zero}_+\{f\}$ must be the inverse of that from a point in $\text{zero}_-\{f\}$. Thus, the general form for the generating functional of an invariant here must have the form $h(z)^{2g-2}$ where $h(z)$ is any formal power series.

The remainder of this section is occupied with the proof of Lemma 6.1.

Proof of Lemma 6.1. There are three steps to the proof. The first step establishes Assertion 1. The second step establishes that these submanifolds are n-non-degenerate and that each $\det(D_\iota)$ has the asserted sign (Assertions 2-4). The third step proves the final assertion of the lemma which concerns the almost complex structures J_1 near to those in (6.3).

Step 1. To begin, let $\{f_m\}$ be a sequence of closed 1-forms on Σ with the following properties:

(6.8)

1. $|f_m| + |\nabla f_m| < m^{-1}$.
2. For each m, there is a connected, pseudo-holomorphic (with J defined using f_m) submanifold $C_m \subset X$ whose fundamental class is Poincaré dual to $n_0 \cdot e$ for some positive $n_0 \leq n$.

The claim is that when m is large, the integer n above must be equal to 1, and C_m has the form $S^1 \times S^1 \times p$, where p is a critical point of f_m. Accept this claim for the moment and it follows directly that there exists $\epsilon > 0$ with the following significance: Suppose that f is a smooth, closed 1-form with non-degenerate zeros and $|f| + |\nabla f| < \epsilon$. Then the connected, pseudo-holomorphic submanifolds of the resulting J have the form $S^1 \times S^1 \times p$ with p being a zero of f. This is Assertion 1 of Lemma 6.1.

With the preceding understood, the remainder of Step 1 is occupied with a proof of the preceding claim.

To start the proof, it is necessary to first study the pseudo-holomorphic curves in the case where $f \equiv 0$. In this case, X is a complex manifold (call it X_0) and the projection of X_0 onto Σ is a holomorphic map. This implies that the holomorphic curves in X_0 with fundamental class Poincaré dual to $n \cdot e_1$ project to points in Σ.

Now consider the sequence $\{C_m\}$ as above. Using known compactness properties of pseudo-holomorphic maps [2], [6] and [15], [5], one can show that there is a subsequence (henceforth renumbered as the original) such that the sequence $\{C_m\}$ converges to a complex submanifold $C \subset X_0$ which is a union of n or less tori, each of the form $S^1 \times S^1 \times$ point. In particular, the maximum distance from points of C_m to C tends to zero as m tends to infinity. Since each C_m is assumed connected, it follows $C = S^1 \times S^1 \times p_0$, where p_0 is a single point in Σ.

Now, the operator D for the submanifold C is simply the operator $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right)$ on $S^1 \times S^1$. Its kernel consists of the constant functions. Thus, one can argue as in the proof of Lemma 5.15 that the Poincaré dual to C_m is equal to e_1.

Now, let $p : X \longrightarrow S^1 \times S^1$ denote the projection onto the obvious factor in (6.1). This map is pseudo-holomorphic (no matter what the choice of f). The restriction of p to C_m is therefore holomorphic, which implies that p is 1 to 1. With this understood, C_m (for m large) can be written as a graph over C. That is, let (x, y) be local coordinates for Σ near p_0 chosen so that $J \cdot \frac{\partial}{\partial x} = \frac{\partial}{\partial y}$ and so that $w|_{p_0} = dx \wedge dy$. Then C_m has the form

$$(6.8) \qquad (t_1, t_2, x(t_1, t_2), y(t_1, t_2)),$$

where x and y are now functions of the variables (t_1, t_2).

The assertion that C_m is pseudo-holomorphic translates into the following system of differential equations for $\eta \equiv x + iy$:

$$(6.9) \qquad \overline{\partial}\eta - \frac{1}{\kappa} f_{\overline{\eta}} = 0,$$

where κ is defined by the identity $w = \kappa \cdot dx \wedge dy$, and $f_{\overline{\eta}}$ is defined by writing $f = f_{\overline{\eta}} d\overline{\eta} + f_\eta d\eta$. One is looking here for a solution of (6.9) with

$|\eta|$ everywhere small. In fact, given $\epsilon > 0$, then the relevant solution to (6.9) will have $|\eta| < \epsilon$ everywhere if m is sufficiently large.

With the preceding understood, the next task is to prove that (6.9) has no non-trivial solutions as described with small norm for η. Indeed, consider first the case where the point p_0 is not a critical point of f. In this case,

$$\frac{1}{\kappa} f_{\overline{\eta}} = c_0 + \mathcal{O}(\epsilon) \tag{6.10}$$

with c_0 being a non-zero constant. Then, integration of both sides of (6.9) over C produces, after an integration by parts, the assertion that the norm of c_0 is $\mathcal{O}(\epsilon)$. Since c_0 is determined by p_0, this is impossible when ϵ is small (hence, when m is large.)

This last argument shows that p_0 must be a critical point of f in order for (6.9) to have an appropriate size solution for all m. In this case, one must replace (6.10) by

$$\frac{1}{\kappa}\frac{\partial f}{\partial \overline{\eta}} = -\nu \cdot \eta - \mu \cdot \overline{\eta} + \mathcal{O}(|\eta|^2). \tag{6.11}$$

Here,

$$\begin{aligned} \nu &= -\frac{\partial f_{\overline{\eta}}}{\partial \eta}\Big|_{\eta=0} \quad \text{and} \\ \mu &= -\frac{\partial f_{\overline{\eta}}}{\partial \overline{\eta}}\Big|_{\eta=0} \end{aligned} \tag{6.12}$$

are determined by f at p_0. Remark now that if there are <u>nontrivial</u> solutions η to (6.9) and (6.11) with arbitrarilly small norm everywhere, then there will be a nontrivial solution to the equation

$$\overline{\partial}\eta + \nu\eta + \mu\overline{\eta} = 0. \tag{6.13}$$

This is just the argument in the proof of Lemma 5.3. However, a corollary of the following lemma is that there are no non-trivial solutions to (6.13) when ν and μ are small and f has non-degenerate critical points.

Lemma 6.3. *Let C be a complex torus given as $\mathbb{C}/H$, where $H \subset \mathbb{C}$ is a lattice generated by $(1, \tau)$ with $\mathrm{im}(\tau) > 0$. Let ν and μ be complex numbers. Then any complex valued function η on T which solves (6.13) is a linear combination of functions of the form:*

$$\eta = \alpha \cdot \exp(i \cdot \mathrm{Re}(\sigma z)) + \beta \cdot \exp(-i \cdot \mathrm{Re}(\sigma z)).$$

Here, σ and α, β are constrained so that:

1. $\mathrm{Re}(\sigma), \mathrm{Re}(\sigma\tau) \in 2\pi\mathbb{Z}$.
2. $|\mu|^2 = -\frac{1}{4}|\sigma|^2 + |\nu|^2$.
3. $\mathrm{Re}(\overline{\sigma}\nu) = 0$.
4. $\overline{\alpha}\left(-\frac{i\cdot\sigma}{2} + \overline{\nu}\right) + \overline{\mu}\beta = 0$.
5. $\beta = \alpha$ *when* $\sigma = 0$.

Proof of Lemma 6.3. This follows by Fourier transforming (6.13).

By Lemma 6.3, for a given complex torus C there are no non-constant solutions to (6.13) when (ν, μ) are small. Furthermore, in the case where ν and μ are both small, the existence of the constant solution requires $|\mu|^2 = |\nu|^2$. In the circumstances of Lemma 6.1, both ν and μ are determined by the 1^{st} order Taylors expansion of the 1-form f at zero p_0, and the condition $|\mu|^2 = |\nu|^2$ is forbidden when p_0 is a non-degenerate zero.

This last assertion proves the claim that for all m sufficiently large, the submanifold C_m must equal $S^1 \times S^1 \times p_0$ where p_0 is a zero of f_m.

Step 2. First of all, one must suppose that $|f| + |\nabla f|$ are small so that all pseudo-holomorphic submanifolds with fundamental class Poincaré dual to $n \cdot e_1$ have $n = 1$ and are of the form $S^1 \times S^1 \times p$, where p is a critical points of f. It then follows immediately from Lemma 6.3 that C is n-non-degenerate, when ν and μ are small, and f has non-degenerate zeros. Here, the maximum size of $|f| + |\nabla f|$ depends on n. Indeed, as explained in Section 5, any section s of some $V_\rho \otimes \mathbb{C}$ in the kernel of D, which is the operator on the left-hand side of (6.13) can be interpreted as a function on some covering torus C' which is annihilated by the operator D for C'. (The operator D on C' is also given by the same left-hand side of (6.13).)

As for sign(det(D)), from the definition, (2.13), and Lemma 6.3 it follows that sign(det(D)) $= 1$ when $|\mu|^2 < |\nu|^2$, and sign(det(D)) $= -1$ when $|\mu|^2 > |\nu|^2$. This is to say that sign(det(D)) $= \pm 1$ depending on whether $p \in \mathrm{zero}_\pm\{f\}$. The fact that sign(det(D_ι)) $= 1$ for non-zero ι follows by a similar analysis.

Step 3. To prove the final part of Lemma 6.1, remark first that the first two conditions of (5.15) have already been established, so it remains only to prove that the third condition is obeyed. With the preceding understood, assume, to the contrary, that this last part of (5.15) is false. This assumption will be seen to yield a contradiction, thus proving the assumption false and the last assertion of Lemma 6.1 true.

To obtain the needed contradiction, note first that under the assumption above, there would exist, for each positive integer m, a function f_m having non-degenerate zeros and with $|f_m| + |\nabla f_m| < m^{-1}$. Furthermore, for each m, the almost complex structure J (as defined using $f \equiv f_m$ in (6.3)) would be a limit of almost complex structures which had more pseudo-holomorphic submanifolds then had $J \equiv J_m$. By the Gromov compactness theorem, as in [6] and [15], and Lemma 5.3 one could then conclude, that there is a complex torus C_m and a J_m-holomorphic map $\varphi_m : C_m \longrightarrow X$ that has the following two properties:

(6.14)

1. φ_m pushes the fundamental class of C_m forward as the Poincaré dual of $n_0 \cdot e_1$ for some positive $n_0 \leq n$. (Cusp curves do not arise here because $n_0 \cdot e_1$ is not a spherical class.)
2. $d\varphi_m = 0$ somewhere.

As remarked, such a pair (C_m, φ_m) would exist for each m, and then the compactness theorem in [6] and [15] could be invoked to conclude that such a pair (C, φ) would exist with φ being J_0-holomorphic. However, all J_0-holomorphic tori factor through $(S^1 \times S^1 \times \{\text{Point}\})$ as covering maps, so a contradiction to the original assumption is obtained.

7. D on other surfaces

This section is a non-sequitor as far as the other sections are concerned in that its purpose is only to illustrate how to use an analog of D in (2.7) to prove the Riemann-Roch theorem on a compact, complex surface. To be precise, suppose that C is a compact, complex surface of genus g, and that $E \longrightarrow C$ is a holomorphic line bundle. According to the Riemann-Roch theorem, the index of the $\overline{\partial}$, as it maps sections of E to sections of $T^{0,1}C \otimes E$, is given by

$$\text{index}(\overline{\partial}) = c_1(E) + 1 - g. \tag{7.1}$$

What follows in this section is a novel (at least to the author) proof of (7.1): Fix a section μ_0 of $T^{1,0}C \otimes E^2$ with non-degenerate zeros. Fix a positive number r and set $\nu \equiv 0$ and $\mu \equiv r \cdot \mu_0$. With these choices understood, use (2.7) to define a differential operator, $D : C^\infty(E) \longrightarrow C^\infty(T^{1,0}C \otimes E)$. Here, D is understood to be $\mathbb{R}$-linear as opposed to being $\mathbb{C}$-linear. As D differs from $\overline{\partial}$ by a zero'th order term, the index of D as a real operator is equal to twice the index of $\overline{\partial}$ as a complex operator.

In order to compute the index of D, it is important to recall that the Euler class of a complex line bundle $F \longrightarrow C$ can be computed by a weighted count (with ± 1 as weights) of the zero's of a section λ of F which vanishes transversely. Indeed, at a zero, x, of F, the differential of λ gives a well defined $\mathbb{R}$-linear homorphism from $TC|_x$ to $F|_x$. If λ vanishes transversely at x, then this homomorphism is an isomorphism. Thus, x contributes $+1$ to the Euler class count when this isomorphism is orientation preserving. Otherwise, x contributes -1.

Now return to the operator D. Take a Riemannian metric on X which is compatible with the complex structure. Also, fix a hermitian structure on E. Now D has a Bochner-Weitzenboch formula whose integral form is as follows:

$$\int_C |Ds|^2 = \int_C \left(|\overline{\partial} s + \nu s|^2 + \mathrm{Re}((\overline{\nu}\mu - \partial\mu)\overline{s}^2) + |\mu|^2|s|^2\right). \tag{7.2}$$

This Bochner-Weitzenboch formula can be used to prove that for large r, the support of any $s \in \mathrm{kernel}(D)$ is concentrated near the set of zeros of μ_0. To be precise here, fix $\epsilon > 0$ and $\rho > 0$. Then, for all r sufficiently large and for any s in the kernel of D (for the given r), all but a fraction less than ϵ of the L^2 norm of s is accounted for by integrating solely over the radius ρ balls about the zeros of μ_0. The point is that the left-hand side of (7.2) is zero, and the right-hand side is $\mathcal{O}(r^2)$ if s has a significant fraction of its support away from the zeros of μ_0. A similar Bochner-Weitzenboch formula holds of D^*. Thus, the support of any $s^* \in \mathrm{cokernel}(D)$ is concentrated near the zeros of μ_0 for large r.

More to the point, with the help of an r-dependent dilation around a zero of μ_0, one can prove that each positive zero contributes, for r sufficiently large, a 1-dimensional subspace to the kernel of D, while each negative zero contributes at large r a 1-dimensional subspace to the cokernel of D. After dilating, the analysis reduces to that for the operator D on $\mathbb{C}$ where $\nu = 0$ and μ is an $\mathbb{R}$-linear isomorphism from $\mathbb{C}$ to $\mathbb{C}$. If the linear map μ here preserves orientation, then D has a one-dimensional L^2-kernel and trivial cokernel. If μ reverses orientation, then the opposite occurs. These remarks are proved with some simple integration by parts tricks. Note that the kernels of D and D^* can be written out explicitly in the case where μ is a linear function on $\mathbb{C}$. For example, if $\mu = \mu_0 \cdot z$, then the L^2 – kernel of D is the real span of $\lambda \cdot e^{-r|z|^2}$ where $\lambda^2 = \mu_0^{-1}$.

Thus, this large r localization of the kernel and cokernel of D to $\mu_0^{-1}(0)$ makes the count for the index of D equal to that for the Euler class of $T^{1,0}C \otimes E^2$. This number is $2 \cdot (c_1(E) + 1 - g)$.

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